Unit 4-7: Infinite Range Ising Model and 1D Ising Model

Finding exact solutions to the Ising model is difficult! But there are two cases in which it is easy!

Infinite Range Ising Model

The mean-field approximate solution turns out to be *exact* in the limit that every spin interacts with every other spin (not just nearest neighbors). Then we have,

$$\mathcal{H} = -\frac{\tilde{J}}{2}\sum_{i,j} s_i s_j - h\sum_i s_i \tag{4.7.1}$$

In the first term, each spin s_i interacts with every other spin s_j, including itself ($i = j$), so there are N^2 terms in the sum.

We can now regroup the double sum on i and j to get

$$\mathcal{H} = -\frac{\tilde{J}}{2}\sum_i s_i\left(\sum_j s_j\right) - h\sum_i s_i = -\frac{\tilde{J}}{2}M\sum_i s_i - h\sum_i s_i = -\left(\frac{\tilde{J}}{2}M + h\right)\sum_i s_i \tag{4.7.2}$$

where $M = \sum_i s_i$ is the total magnetization. Note, since the first term involves a sum over N^2 terms, if we want the energy of the system to be extensive, i.e. $E \propto N$, it is necessary that the coupling scale as $\tilde{J} \propto 1/N$. We therefore define $\tilde{J} = zJ/2N$ and $m = M/N$. Note, here $zJ/2$ is just a name we give to the coupling per pair of spins so that things will look more like what we had in the mean-field approximate solution; we can think of z as being the number of nearest neighbor spins if we like, but that number has no special significance in this model since each spin interacts with every other spin no matter how far away.

With this the above becomes,

$$\mathcal{H} = -\left(\frac{zJ}{4}m + h\right)\sum_i s_i \tag{4.7.3}$$

which looks just like the mean-field Hamiltonian in that each s_i seems decoupled from all the other s_j. However, there is an important difference from the mean-field model because the quantity m that appears in the above expression varies from configuration to configuration, since it is $m = (1/N)\sum_i s_i$, and $\sum_i s_i$ is not fixed. Since m depends on the $\{s_i\}$, there remains an effective coupling between the spins, and we cannot simply do the partition function sum over the $\{s_i\}$ like we do for non-interacting spins in the mean-field model.

We therefore try a different approach. Instead of working in the constant h Gibbs ensemble, let us work in the constant $M = Nm$ Helmholtz ensemble. Then the Hamiltonian is simply,

$$\mathcal{H} = -\frac{\tilde{J}}{2}\sum_{i,j} s_i s_j = -\frac{\tilde{J}}{2}\left(\sum_i s_i\right)\left(\sum_j s_j\right) = -\frac{\tilde{J}}{2}M^2 = -\frac{zJ}{4N}M^2 \tag{4.7.4}$$

The partition function is then,

$$\tilde{Z}(M,T) = \sum_{\{s_i\}} \delta\left(\sum_i s_i - M\right) \mathrm{e}^{-\beta\mathcal{H}} = \Omega(M)\, \mathrm{e}^{\beta zJM^2/4N} \tag{4.7.5}$$

The delta function imposes the constraint that we sum only over configurations in which the total magnetization is fixed to the value M. But since the Boltzmann factor for such configurations are all the same (i.e. they only depend on M and not the individual s_i), the partition function is just the Boltzmann factor times the number of configurations $\Omega(M)$ that have the specified M.

We can find $\Omega(M)$ like we did in an early homework problem! If N_+ is the number of spins with $s_i = +1$, and N_- is the number of spins with $s_i = -1$, then we have,

$$\left.\begin{aligned} N_+ + N_- &= N \\ \\ N_+ - N_- &= M \end{aligned}\right\} \qquad \Rightarrow \qquad \left\{\begin{aligned} N_+ &= \frac{N+M}{2} \\ N_- &= \frac{N-M}{2} \end{aligned}\right. \tag{4.7.6}$$

and

$$\Omega(M) = \frac{N!}{N_+!\,N_-!} = \frac{N!}{\left(\frac{N+M}{2}\right)!\left(\frac{N-M}{2}\right)!} \tag{4.7.7}$$

We then have for the Helmholtz free energy,

$$F(M,T) = -k_B T\left[\ln\Omega(M) + \frac{\beta z J}{4N}M^2\right] = -\frac{zJ}{4N}M^2 - k_B T\left[N\ln N - N_+\ln N_+ - N_-\ln N_-\right] \tag{4.7.8}$$

where we used the Stirling formula for $\ln N!$ and that $N = N_+ + N_-$. We then have for the magnetic field,

$$h = \left(\frac{\partial F}{\partial M}\right)_T = -\frac{zJ}{2N}M + k_B T\left[\left(\frac{\partial N_+}{\partial M}\right)\ln N_+ + \frac{N_+}{N_+}\left(\frac{\partial N_+}{\partial M}\right) + \left(\frac{\partial N_-}{\partial M}\right)\ln N_- + \frac{N_-}{N_-}\left(\frac{\partial N_-}{\partial M}\right)\right] \tag{4.7.9}$$

using $\partial N_+/\partial M = +1/2$ and $\partial N_-/\partial M = -1/2$ we then have,

$$h = -\frac{zJ}{2N}M + \frac{k_B T}{2}\ln\left[\frac{N_+}{N_-}\right] = -\frac{zJ}{2N}M + \frac{k_B T}{2}\ln\left[\frac{N+M}{N-M}\right] = -\frac{zJm}{2} + \frac{k_B T}{2}\ln\left[\frac{1+m}{1-m}\right] \tag{4.7.10}$$

where $m = M/N$ is the magnetization density. Continuing,

$$h = -\frac{zJm}{2} + \frac{k_B T}{2}\left[\ln(1+m) - \ln(1-m)\right] \tag{4.7.11}$$

Near T_c we have m is small, so we can expand $\ln(1\pm m)$ to get,

$$h = -\frac{zJm}{2} + \frac{k_B T}{2}\left[\left(m - \frac{m^2}{2} + \frac{m^3}{3} + \cdots\right) - \left(-m - \frac{m^2}{2} - \frac{m^3}{3} + \cdots\right)\right] \tag{4.7.12}$$

$$= -\frac{zJm}{2} + k_B T\left[m + \frac{1}{3}m^3\right] = k_B T\left[\left(1 - \frac{zJ}{2k_B T}\right)m + \frac{1}{3}m^3\right] \tag{4.7.13}$$

This is *exactly* the same expression we found from the mean-field theory calculation in Eq. (4.4.7), if we identify $T_c = zJ/2k_B$. Since the equation of state, $h(m,T)$, is the same as in mean-field theory, it follows that all the critical exponents must be the same!

We can even do better than the expansion for small m. From Eq. (4.7.11) we can write,

$$\tanh\left(\frac{\beta z J m}{2} + \beta h\right) = \tanh\left[\frac{1}{2}\ln\left(\frac{1+m}{1-m}\right)\right] = \tanh\left[\ln\left(\frac{1+m}{1-m}\right)^{1/2}\right] \tag{4.7.14}$$

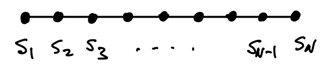
$$= \frac{\sqrt{\dfrac{1+m}{1-m}} - \sqrt{\dfrac{1-m}{1+m}}}{\sqrt{\dfrac{1+m}{1-m}} + \sqrt{\dfrac{1-m}{1+m}}} = \frac{1 - \dfrac{1-m}{1+m}}{1 + \dfrac{1-m}{1+m}} = \frac{2m}{2} = m \tag{4.7.15}$$

So we get

$$m = \tanh\left(\frac{\beta z J m}{2} + \beta h\right) \tag{4.7.16}$$

which is then equal to the mean-field result of Eq. (4.4.1).

Ising Model in One Dimension with $h = 0$



For simplicity we will start with the case $h = 0$. In one dimension, the spins form a linear chain, where s_i interacts only with s_{i+1} and s_{i-1}, as in the sketch on the right. For a finite chain of N spins, we will take *free boundary conditions* where s_1 interacts only with s_2, and where s_N interacts only with s_{N-1}. The Hamiltonian is then,

$$\mathcal{H} = -J \sum_{i=1}^{N-1} s_i s_{i+1} \tag{4.7.17}$$

Define,

$$\sigma_i \equiv s_i s_{i+1}, \quad i = 1, 2, \ldots, N-1 \qquad \sigma_i = \pm 1 \tag{4.7.18}$$

Then the Hamiltonian can be written as,

$$\mathcal{H} = -J \sum_{i=1}^{N-1} \sigma_i \tag{4.7.19}$$

We have

$$\prod_{i=1}^{j-1} \sigma_i = \sigma_1 \sigma_2 \cdots \sigma_{j-1} = (s_1 s_2)(s_2 s_3) \cdots (s_{j-1} s_j) = s_1 (s_2)^2 (s_3)^2 \cdots (s_{j-1})^2 s_j = s_1 s_j \tag{4.7.20}$$

So if we know s_1 and all the σ_i, we can uniquely reconstruct the spins

$$s_j = \frac{1}{s_1} \prod_{i=1}^{j-1} \sigma_i . \tag{4.7.21}$$

We can therefore write for the partition function,

$$Z = \sum_{\{s_i\}} \mathrm{e}^{-\beta \mathcal{H}} = \sum_{\{s_i\}} \mathrm{e}^{\beta J \sum_{i=1}^{N-1} s_i s_{i+1}} = \sum_{s_1} \sum_{\{\sigma_j\}} \mathrm{e}^{\beta J \sum_{j=1}^{N-1} \sigma_j} = 2 \sum_{\{\sigma_j\}} \mathrm{e}^{\beta J \sum_{j=1}^{N-1} \sigma_j} \tag{4.7.22}$$

Since the Boltzmann factors are independent of s_1, the sum on $s_1 = \pm 1$ just gives a multiplicative factor of 2. Writing the exponential of a sum as the product of exponentials, we then get,

$$Z = 2 \sum_{\{\sigma_j\}} \left(\prod_{j=1}^{N-1} \mathrm{e}^{\beta J \sigma_j} \right) = 2 \prod_{j=1}^{N-1} \left(\sum_{\sigma_j = \pm 1} \mathrm{e}^{\beta J \sigma_j} \right) = 2 \prod_{j=1}^{N-1} \left(\mathrm{e}^{\beta J} + \mathrm{e}^{-\beta J} \right) = 2 \left(2 \cosh \beta J \right)^{N-1} \tag{4.7.23}$$

The Gibbs free energy is then

$$G(h = 0, T) = -k_B T \ln Z = -k_B T \ln 2 - k_B T (N-1) \ln \left[2 \cosh \beta J \right] \tag{4.7.24}$$

The Gibbs free energy per spin in the $N \to \infty$ thermodynamic limit is then,

$$\boxed{g(h = 0, T) = \lim_{N \to \infty} \frac{G}{N} = -k_B T \ln \left[2 \cosh \beta J \right]} \tag{4.7.25}$$

The entropy per particle is then

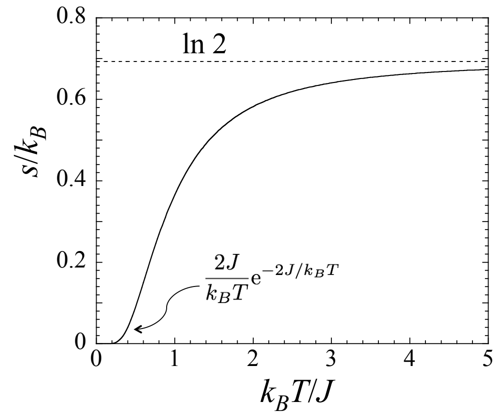
$$s = -\left(\frac{\partial g}{\partial T} \right)_{h=0} = k_B \ln \left[2 \cosh \beta J \right] + \frac{2 k_B T}{2 \cosh \beta J} \frac{\partial}{\partial T} (\cosh \beta J) \tag{4.7.26}$$

$$= k_B \ln \left[2 \cosh \beta J \right] + \frac{2 k_B T}{2 \cosh \beta J} (\sinh \beta J) J \frac{d\beta}{dT} \tag{4.7.27}$$

$$= k_B \ln \left[2 \cosh \beta J \right] - \frac{J}{T} \tanh \beta J \tag{4.7.28}$$

$$\boxed{s = k_B\left[\ln(2\cosh\beta J) - \beta J\tanh\beta J\right]} \tag{4.7.29}$$

We now check the high and low temperature limits.



As $T \to \infty$, $\beta \to 0$, $\cosh\beta J \approx 1 + \frac{1}{2}(\beta J)^2$, $\tanh\beta J \approx \beta J$, so,

$$s \approx k_B\left[\ln[2 + (\beta J)^2] - (\beta J)^2\right] \approx k_B \ln 2 \tag{4.7.30}$$

This is as expected: at very large T each spin is equally likely to be up as down, so there are 2^N equally likely states, and so $S = k_B \ln 2^N = k_B N \ln 2$ and $s = S/N = k_B \ln 2$.

As $T \to 0$, $\beta \to \infty$, $\cosh\beta J \approx \frac{1}{2}\mathrm{e}^{\beta J}$, $\tanh\beta J = \dfrac{1 - \mathrm{e}^{-2\beta J}}{1 + \mathrm{e}^{-2\beta J}} \approx 1 - 2\mathrm{e}^{-2\beta J}$, so,

$$s \approx k_B\left[\ln \mathrm{e}^{\beta J} - \beta J\left(1 - 2\mathrm{e}^{-2\beta J}\right)\right] \approx \frac{2J}{T}\mathrm{e}^{-2J/k_BT} \tag{4.7.31}$$

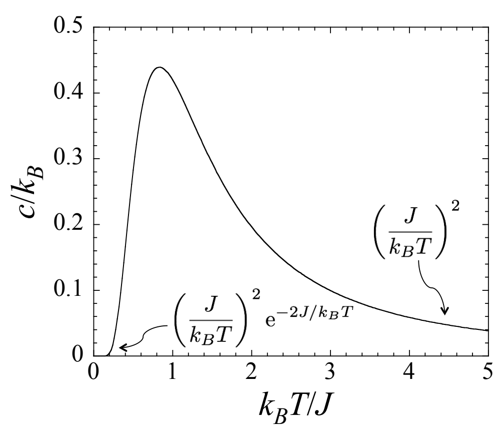
We see that $s(h = 0, T)$ has an essential singularity $\sim \mathrm{e}^{-2J/k_BT}$ at $T = 0$. An essential singularity means that no Taylor series exists at $T = 0$ with a finite radius of convergence.

Next we consider the specific heat per spin at constant $h = 0$

$$c = T\left(\frac{\partial s}{\partial T}\right)_{h=0} = k_BT\left[\frac{-2J\sinh\beta J}{2\cosh\beta J}\frac{1}{k_BT^2} + \frac{J}{k_BT^2}\tanh\beta J + \frac{\beta J^2}{k_BT^2}\frac{\partial}{\partial(\beta J)}\tanh\beta J\right] \tag{4.7.32}$$

$$= \frac{J^2}{k_BT^2}\frac{\partial}{\partial(\beta J)}\tanh\beta J = \frac{J^2}{k_BT^2}\frac{1}{(\cosh\beta J)^2} \tag{4.7.33}$$

$$\boxed{c = k_B\left(\frac{\beta J}{\cosh\beta J}\right)^2} \tag{4.7.34}$$



As $T \to \infty$, $\beta \to 0$ and $\cosh(\beta J) \to 1$, so we have $c \approx k_B\left(\dfrac{J}{k_BT}\right)^2$.

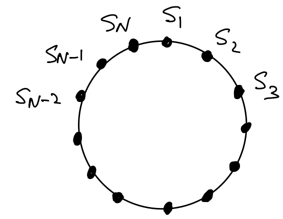
As $T \to 0$, $\beta \to \infty$ and $\cosh(\beta J) \to \mathrm{e}^{\beta J}$, so we then have $c \approx k_B\left(\dfrac{J}{k_BT}\right)^2\mathrm{e}^{-2J/k_BT}$. There is an essential singularity at $T = 0$.

We see that there is no singularity in any thermodynamic quantity at any *finite* T. Thus there is no phase transition at any finite T. The 1D Ising model is always paramagnetic at any finite T.

Why does the 1D Ising model have no ferromagnetic phase transition? The answer is given in Notes 4-2 near the top of page 3, where we discuss the critical domain in 1D.

1D Ising Model in a Magnetic Field, $h > 0$

We will now consider the case where $h > 0$ is finite. The approach of the calculation is entirely different from the $h = 0$ case, because the coupling of the spins to h means that the trick of transforming $\{s_i\} \to \{\sigma_i\}$ no longer simplifies things!



Here we will take the spins to have *periodic boundary conditions*, so that s_1 interacts with s_N. One can think of the spins as lying on the circumference of a circle, as in the sketch to the right. The Hamiltonian is then,

$$\mathcal{H} = -J\left(s_1 s_2 + s_2 s_3 + \cdots + s_{N-1}s_N + s_N s_1\right) - h(s_1 + s_2 + \cdots + s_{N-1} + s_N) \tag{4.7.35}$$

The Boltzmann factor for a given configuration $\{s_i\}$ can then be written as,

$$\mathrm{e}^{-\beta\mathcal{H}} = \mathrm{e}^{\beta J(s_1s_2+s_2s_3+\cdots+s_{N-1}s_N+s_Ns_1)+\beta h(s_1+s_2+\cdots+s_{N-1}+s_N)} \tag{4.7.36}$$

$$= \left[\mathrm{e}^{\beta J(s_1s_2)+\frac{1}{2}\beta h(s_1+s_2)}\right]\left[\mathrm{e}^{\beta J(s_2s_3)+\frac{1}{2}\beta h(s_2+s_3)}\right]\cdots\left[\mathrm{e}^{\beta J(s_{N-1}s_N)+\frac{1}{2}\beta h(s_{N-1}+s_N)}\right]\left[\mathrm{e}^{\beta J(s_Ns_1)+\frac{1}{2}\beta h(s_N+s_1)}\right] \tag{4.7.37}$$

$$= f(s_1,s_2)f(s_2,s_3)\cdots f(s_{N-1},s_N)f(s_N,s_1) \tag{4.7.38}$$

where

$$f(s,s') = \mathrm{e}^{\beta Jss'+\frac{1}{2}\beta h(s+s')} \tag{4.7.39}$$

Since s can take only the two values $s = \pm 1$, we can regard $f(s,s')$ as a 2×2 matrix, known as the *transfer matrix*,

$$f(s,s') = \begin{bmatrix} f(1,1) & f(1,-1) \\ f(-1,1) & f(-1,-1)\end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\beta(J+h)} & \mathrm{e}^{-\beta J} \\ \mathrm{e}^{-\beta J} & \mathrm{e}^{\beta(J-h)}\end{bmatrix} = \mathbb{M}_{ss'} \quad \text{to write it in a matrix notation.} \tag{4.7.40}$$

The partition function sum is then,

$$Z(h,T) = \sum_{\{s_i\}}\mathrm{e}^{-\beta\mathcal{H}} = \sum_{s_1}\sum_{s_2}\cdots\sum_{s_N} f(s_1,s_2)f(s_2,s_3)\cdots f(s_{N-1},s_N)f(s_N,s_1) \tag{4.7.41}$$

$$= \sum_{s_1}\sum_{s_2}\cdots\sum_{s_N}\mathbb{M}_{s_1s_2}\mathbb{M}_{s_2s_3}\cdots\mathbb{M}_{s_{N-1}s_N}\mathbb{M}_{s_Ns_1} \tag{4.7.42}$$

Consider, for example, the sum on s_i. This involves only the terms $f(s_{i-1},s_i)$ and $f(s_i,s_{i+1})$. We see that this sum can be expressed in terms of matrix multiplication,

$$\sum_{s_i} f(s_{i-1},s_i)f(s_i,s_{i+1}) = \sum_{s_i}\mathbb{M}_{s_{i-1}s_i}\mathbb{M}_{s_is_{i+1}} = \left[\mathbb{M}^2\right]_{s_{i-1}s_{i+1}} \tag{4.7.43}$$

In a similar way we get,

$$Z = \sum_{s_1}\left[\mathbb{M}^N\right]_{s_1s_1} = \mathrm{Tr}\left[\mathbb{M}^N\right] \tag{4.7.44}$$

Now, as we will soon show, the matrix $\mathbb{M}$ is diagonable with two real eigenvalues λ_1 and λ_2. Assume λ_1 is the bigger of the two. The trace of a matrix is independent of the basis in which the matrix is expressed. So if we choose the basis of eigenvectors of $\mathbb{M}$, in which the representation of $\mathbb{M}$ is diagonal, we have,

$$Z = \mathrm{Tr}\left[\mathbb{M}^N\right] = \lambda_1^N + \lambda_2^N = \lambda_1^N\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right] \quad \to \quad \lambda_1^N \quad \text{as } N\to\infty. \tag{4.7.45}$$

So now we need to find the eigenvalues of $\mathbb{M}$. These are given by the characteristic equation,

$$\left(e^{\beta(J+h)}-\lambda\right)\left(e^{\beta(J-h)}-\lambda\right)-\left(e^{-\beta J}\right)\left(e^{-\beta J}\right)=0 \tag{4.7.46}$$

$$\Rightarrow \quad \lambda^2-2e^{\beta J}\cosh h\beta J+2\sinh 2\beta J=0 \tag{4.7.47}$$

The largest eigenvalue is then obtained from the positive root of the quadratic formula,

$$\lambda_1=e^{\beta J}\cosh\beta h+\sqrt{e^{2\beta J}\cosh^2\beta h-2\sinh 2\beta J} \tag{4.7.48}$$

With a little algebra, and using $\cosh^2 x-\sinh^2 x=1$, we can rewrite the above as,

$$\lambda_1=e^{\beta J}\left[\cosh\beta h+\sqrt{e^{-4\beta J}+\sinh^2\beta h}\,\right] \tag{4.7.49}$$

The Gibbs free energy density is then,

$$g(h,T)=\frac{G(h,T)}{N}=-\frac{k_BT}{N}\ln Z=-\frac{k_BT}{N}\ln\lambda_1^N=-k_BT\ln\lambda_1 \tag{4.7.50}$$

$$=-k_BT\left(\beta J+\ln\left[\cosh\beta h+\sqrt{e^{-4\beta J}+\sinh^2\beta h}\,\right]\right) \tag{4.7.51}$$

$$=-J-k_BT\ln\left[\cosh\beta h+\sqrt{e^{-4\beta J}+\sinh^2\beta h}\,\right] \tag{4.7.52}$$

Note, if we take $h=0$, then we get from Eqs. (4.7.49) and (4.7.50),

$$g(h=0,T)=-k_BT\ln\lambda_1=-k_BT\ln\left[e^{\beta J}\left(1+\sqrt{e^{-4\beta J}}\right)\right]=-k_BT\ln\left[e^{\beta J}+e^{-\beta J}\right] \tag{4.7.53}$$

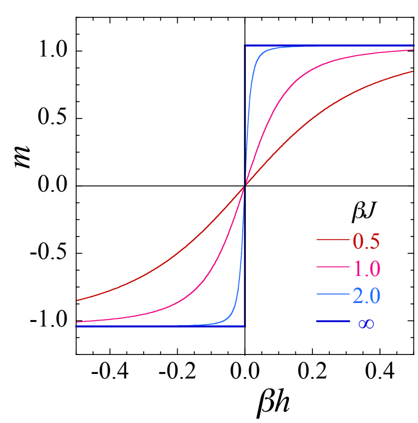
$$=-k_BT\ln\left[2\cosh\beta J\right] \tag{4.7.54}$$

which agrees with our result of Eq. (4.7.25) for the direct $h=0$ calculation. We will thus find, for $h=0$, the same entropy s and specific heat c as we found before.

But now we can also find the magnetization m and the magnetic susceptibility χ. We have,

$$m(h,T)=-\left(\frac{\partial g}{\partial h}\right)_T=\left[\frac{\sinh\beta h+\dfrac{\sinh\beta h\cosh\beta h}{\sqrt{e^{-4\beta J}+\sinh^2\beta h}}}{\cosh\beta h+\sqrt{e^{-4\beta J}+\sinh^2\beta h}}\right] \tag{4.7.55}$$

$$=\frac{\sinh\beta h}{\sqrt{e^{-4\beta J}+\sinh^2\beta h}} \tag{4.7.56}$$



Note that when $h\to 0$, we have $m\to 0$ at any finite β, so there is no spontaneous magnetization at any finite temperature T. There is no finite temperature ferromagnetic phase transition.

However, if we look at the limit $\beta\to\infty$, i.e. $T\to 0$, we get,

$$m(h,T\to 0)=\frac{\sinh\beta h}{\pm\sinh\beta h}=\pm 1 \tag{4.7.57}$$

So the spins become fully magnetized, with all $s_i = \pm 1$, at $T = 0$. There is thus a ferromagnetic phase transition at $T = 0$, with a discontinuous jump in m from $m = 0$ at $T = 0^+$ to $m = \pm 1$ at $T = 0$.

In the sketch above to the right we plot m vs h for a few values of $\beta J = J/k_B T$. We see that for finite β, m is always continuous as h passes through zero, thus demonstrating that there is no spontaneous magnetization. Only at $\beta J \to 0$, i.e. $T \to 0$, do we have a discontinuous jump in m.

Finally we can compute the magnetic susceptibility,

$$\chi = \lim_{h\to 0}\left(\frac{\partial m}{\partial h}\right)_T = \lim_{h\to 0}\left[\frac{\beta\cosh\beta h}{\sqrt{\mathrm{e}^{-4\beta J} + \sinh^2\beta h}} - \frac{\sinh^2\beta h\ \cosh\beta h}{\left(\mathrm{e}^{-4\beta J} + \sinh^2\beta h\right)^{3/2}}\right] = \frac{\mathrm{e}^{2J/k_BT}}{k_BT} \tag{4.7.58}$$

We thus see that χ is finite at any finite temperature T, and diverges only as $T \to 0$. This again indicates that there is no finite temperature phase transition.