Solutions Problem Set 2

1) From the ideal gas law

$$pV = Nk_B T$$

we have for an isotherm, ie T held constant

$$pV = \text{const} \Rightarrow \boxed{p \sim \frac{1}{V} \text{ on an isotherm}}$$

For an adeabat, ie S held const, we use from Problem (1)

$$E(S,V,N) = E_0\left(\frac{V}{V_0}\right)^{-2/3}\left(\frac{N}{N_0}\right)^{5/3}\exp\left[\frac{2}{3}\left(\frac{S}{N}-\frac{S_0}{N_0}\right)/k_B\right]$$

$$p = -\left(\frac{\partial E}{\partial V}\right)_{S,N} = \frac{2}{3}\frac{E_0}{V_0}\left(\frac{V}{V_0}\right)^{-5/3}\left(\frac{N}{N_0}\right)\exp\left[\frac{2}{3}\left(\frac{S}{N}-\frac{S_0}{N_0}\right)/k_B\right]$$

so when S and N are held const

$$pV^{5/3} = \text{const} \Rightarrow \boxed{p \sim \frac{1}{V^{5/3}} \text{ on an adiabat}}$$

or $V \sim \frac{1}{p}$ on an isotherm (const T)

$V \sim \frac{1}{p^{3/5}}$ on an adeabat (const S)

2)

Prove $\left(\frac{\partial C_p}{\partial p}\right)_T = -TV\left[\alpha^2 + \left(\frac{\partial \alpha}{\partial T}\right)_p\right]$

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_p \quad = \text{definition of } C_p$$

$$S(T,p) = -\left(\frac{\partial G(T,p)}{\partial T}\right)_p$$

$$C_p = -T\left(\frac{\partial^2 G}{\partial T^2}\right)_p$$

$$\left(\frac{\partial C_p}{\partial p}\right)_T = -T\left(\frac{\partial^3 G}{\partial T^2 \partial p}\right) = -T\left(\frac{\partial^2}{\partial T^2}\left(\frac{\partial G}{\partial p}\right)_T\right)_p$$

$$= -T\left(\frac{\partial^2 V}{\partial T^2}\right)_p = -T\left(\frac{\partial}{\partial T}\left(\frac{\partial V}{\partial T}\right)_p\right)_p$$

$$= -T\frac{\partial}{\partial T}(V\alpha)_p \quad \text{- using definition of } \alpha$$

$$= -TV\left(\frac{\partial \alpha}{\partial T}\right)_p - T\alpha\left(\frac{\partial V}{\partial T}\right)_p$$

$$= -TV\left(\frac{\partial \alpha}{\partial T}\right)_p - T\alpha V\alpha$$

$$\boxed{\left(\frac{\partial C_p}{\partial p}\right)_T = -TV\left[\alpha^2 + \left(\frac{\partial \alpha}{\partial T}\right)_p\right]}$$

3) $T = \left(\frac{V}{V_0}\right)^{\eta} T_0$ gives temperature as system changes volume V

a) Work done on the gas

$$dW = -p\,dV$$

use ideal gas $pV = Nk_BT \Rightarrow p = \frac{Nk_BT}{V}$

$$dW = -Nk_BT\frac{dV}{V}$$

total work done is

$$W = \int dW = -\cancel{N}k_B \int_{V_0}^{V_1} \frac{dV}{V}\,T(V)$$

$$= Nk_B \int_{V_0}^{V_1} \frac{dV}{V}\left(\frac{V}{V_0}\right)^{\eta} T_0 = -Nk_BT_0 \int_{V_0}^{V_1} dV\,\frac{V^{\eta-1}}{V_0^{\eta}}$$

$$= \frac{-Nk_BT_0}{V_0^{\eta}}\,\frac{V_1^{\eta}-V_0^{\eta}}{\eta} = \frac{Nk_BT_0}{\eta}\left(1-\left(\frac{V_1}{V_0}\right)^{\eta}\right)$$

$$\boxed{W = \frac{Nk_BT_0}{\eta}\left[1-\left(\frac{V_1}{V_0}\right)^{\eta}\right]}$$

b) use $E = \frac{3}{2}Nk_BT$ for ideal gas

$$\boxed{\Delta E = \frac{3}{2}Nk_B(T_1 - T_0) = \frac{3}{2}Nk_BT_0\left[\left(\frac{V_1}{V_0}\right)^{\eta}-1\right]}$$

c) Heat transfer to the gas

$$dQ \equiv TdS = dE - dW \qquad \left(\text{from } dE = TdS - p\,dV\right.$$
$$\text{and } dW = -p\,dV\Big)$$

$$\Rightarrow \Delta Q = \Delta E - \Delta W$$

$$\Delta Q = N k_B T_0 \left[\left(\frac{V_1}{V_0}\right)^{\eta} - 1\right]\frac{3}{2} - \frac{N k_B T_0}{\eta}\left[1 - \left(\frac{V_1}{V_0}\right)^{\eta}\right]$$

$$\boxed{\Delta Q = N k_B T_0 \left[\left(\frac{V_1}{V_0}\right)^{\eta} - 1\right]\left[\frac{3}{2} + \frac{1}{\eta}\right]}$$

$\Delta Q = 0$ when $\frac{3}{2} + \frac{1}{\eta} = 0 \Rightarrow \boxed{\eta = -\frac{2}{3}}$

From problem 1 if the gas expands adiabatically (ie $\Delta S = 0 \Rightarrow \Delta Q = 0$) then

$$p V^{5/3} = \text{const}$$

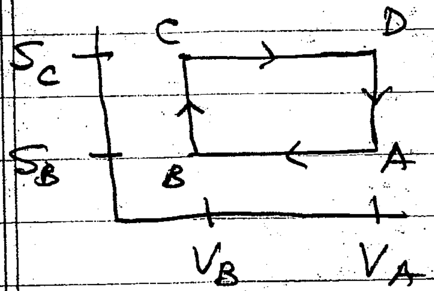
using $pV = N k_B T$ the above becomes

$$\frac{N k_B T}{V} V^{5/3} = N k_B T\, V^{2/3} = \text{const}$$

$$\text{or} \quad T \sim V^{-2/3}$$

which gives $\eta = -2/3$ in agreement with part (c)

4) Otto cycle



heat extracted in $B \to C$
work done by system in $C \to D$
heat returned in $D \to A$
work done on system in $A \to B$

efficiency $\quad \epsilon \equiv \dfrac{Q_{BC} - Q_{AD}}{Q_{BC}} = 1 - \dfrac{Q_{AD}}{Q_{BC}}$

using $dQ = TdS$, the heat extracted from reservoir and added to the system in step $B \to C$ is

$$Q_{BC} = \int_B^C T\,dS$$ integral is taken at constant V

Now we know $\quad T = \left(\dfrac{\partial E}{\partial S}\right)_{V,N}$ so

$$Q_{BC} = \int_B^C \left(\frac{\partial E}{\partial S}\right)_{V,N} dS = E_C - E_B$$

(we can do the integral since trajectory of integration is at constant V, or more generally, $TdS = dE + p\,dV$, but here $dV = 0$)

For ideal gas, as assumed here, $E = \frac{3}{2} N k_B T$

So $Q_{BC} = \frac{3}{2} N k_B (T_C - T_B)$

similarly

$$Q_{AD} = \frac{3}{2} N k_B (T_D - T_A)$$

efficiency

$$\epsilon = 1 - \left(\frac{T_D - T_A}{T_C - T_B}\right)$$

Finally we need to relate T_A, T_B, T_C, T_D to the volumes V_A and V_B

From problem (1) we have for the adiabatic expansion of the ideal gas,

$$p V^{5/3} = \text{const}$$

using $pV = N k_B T$ gives $N k_B T V^{2/3} = \text{const}$

So along path $C \to D$ we get

$$T_C V_B^{2/3} = T_D V_A^{2/3} \Rightarrow T_D = T_C \left(\frac{V_B}{V_A}\right)^{2/3}$$

and along path $B \to A$ we get

$$T_B V_B^{2/3} = T_A V_A^{2/3} \Rightarrow T_A = T_B \left(\frac{V_B}{V_A}\right)^{2/3}$$

So

$$\epsilon = 1 - \left[\frac{T_C \left(\frac{V_B}{V_A}\right)^{2/3} - T_B \left(\frac{V_B}{V_A}\right)^{2/3}}{T_C - T_B}\right]$$

$$\epsilon = 1 - \left(\frac{V_B}{V_A}\right)^{2/3}$$

Now for ideal gas

$$C_V = \frac{3}{2} N k_B$$

$$C_p = \frac{5}{2} N k_B$$

$$\frac{C_p - C_V}{C_V} = \frac{5-3}{3} = \frac{2}{3}$$

$$\boxed{e = 1 - \left(\frac{V_B}{V_A}\right)^{(C_p - C_V)/C_V}}$$