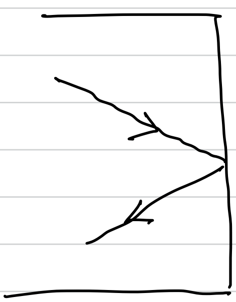
Solutions Problem Set 3

1) N particles in a box of volume V travel with extremely relativistic speeds such that we have $|\vec{v}| \approx c$ and their energy can be taken to be $\varepsilon = pc$, with p the magnitude of the relativistic momentum.

Consider such a particle colliding with the wall. We assume the collision is elastic.



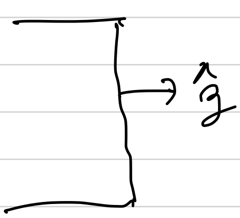
Then, just like in notes 2-1 for non-relativistically moving particles, we can write the pressure on the wall as

$$\mathbb{P} = \left\langle \frac{(\Delta p_\perp)\,\text{rate}}{A} \right\rangle \qquad \left(\begin{array}{l} \mathbb{P} \text{ is pressure} \\ p \text{ is momentum} \end{array} \right.$$

where $\Delta p_\perp = 2p_\perp$ is the change in momentum of the particle during the collision, $p_\perp$ is the component of momentum in the direction perpendicular to the wall, A is the wall area, and the rate of collisions is

$$\text{rate} = \frac{1}{2}\frac{N}{V} A\, v_\perp$$

with $v_\perp$ the component of the velocity in the direction perpendicular to the wall.



If the normal vector to the wall is $\hat{z}$, then $p_\perp = p_z$ and $v_\perp = v_z$

$\langle \cdots \rangle$ represents the average over all particles and all collisions

$$\Rightarrow \mathbb{P} = \frac{N}{V} \langle p_\perp v_\perp \rangle$$

Now, by our assumption that particles travel equally likely in all directions we can write

$$\langle p_\perp v_\perp \rangle = \langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_z \rangle$$
$$= \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle = \frac{1}{3} \langle p v \rangle$$

The last step, $\vec{p} \cdot \vec{v} = pv$, is because $\vec{p}$ and $\vec{v}$ are colinear. Now use $v \simeq c$ and we get

$$\mathbb{P} = \frac{1}{3} \frac{N}{V} \langle p c \rangle = \frac{1}{3} \frac{N}{V} \langle \varepsilon \rangle$$

$$\Rightarrow \langle \varepsilon \rangle = \frac{3 \mathbb{P} V}{N}$$

Now use the ideal gas law $\mathbb{P} V = N k_B T$ to get

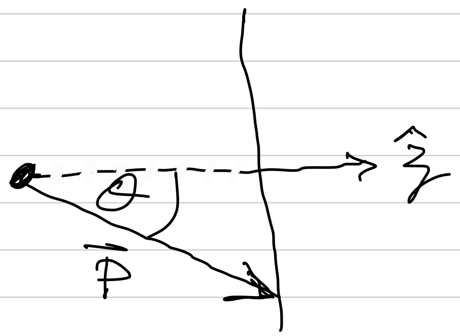
$$\boxed{\langle \varepsilon \rangle = 3 k_B T}$$

compare that to the non-relativistic result $\langle \varepsilon \rangle = \frac{3}{2} k_B T$

To further convince you that

$$\langle p_\perp v_\perp \rangle = \frac{1}{3}\langle p v \rangle$$

we can write



$$p_\perp = p\cos\theta$$
$$v_\perp = v\cos\theta$$

$$\langle p_\perp v_\perp \rangle = \langle p v \cos^2\theta \rangle$$

The average $\langle \cdots \rangle$ is an average over all particle speeds v, and all velocity directions, specified by the spherical angles (φ, θ).

We can first do the average over all directions to get

$$\langle \cos^2\theta \rangle = \underbrace{\frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta}_{\text{average over all spherical angles}} \cos^2\theta$$

$$\langle \cos^2\theta \rangle = \frac{1}{2}\left[\frac{-\cos^3\theta}{3}\right]_0^{\pi} = \frac{1}{2}\left[\frac{1}{3} + \frac{1}{3}\right] = \frac{1}{3}$$

so $\langle p_\perp v_\perp \rangle = \langle p v \cos^2\theta \rangle = \frac{1}{3}\langle p v \rangle$

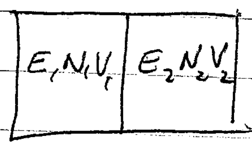
Note: Nowhere in our calculation did we need to use the relativistic expression for momentum

$p = m\gamma v$ with $\gamma = \dfrac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$

2) For an ideal gas, we found that the number of states depended on the total energy E as

$$\Omega(E) \propto E^{\frac{3N}{2}-1} = E^{\gamma} \quad \text{with } \gamma = \frac{3N}{2}-1$$

Consider a box of volume V, split in half. Each half contains equal amounts of the same ideal gas. The wall is thermally conducting



a) $V_1 = V_2$ since box split in half
$T_1 = T_2$ since wall is thermally conducting
$N_1 = N_2$ since equal amounts of gas

ideal gas law $\Rightarrow$ $p_1 = \dfrac{N_1 k_B T_1}{V_1} = \dfrac{N_2 k_B T_2}{V_2} = p_2$

so pressures are equal

b) The number of states of the total system in which the left side has energy E and the right side has energy $E_T - E$ was given in lecture as

$$\Omega(E)\,\Omega(E_T - E)$$

where in this case it is the same function

$\Omega(E)$ for both sides of the box since both sides have the same type of gas and $N_1 = N_2$

The most probable E is given by the value of E that has the largest number of states, ie.

$$\frac{d}{dE}\left[\Omega(E)\,\Omega(E_T - E)\right] = 0$$

$$\Rightarrow \frac{d}{dE}\left[E^{\gamma}(E_T - E)^{\gamma}\right] = 0$$

$$\Rightarrow \gamma E^{\gamma-1}(E_T - E)^{\gamma} - \gamma E^{\gamma}(E_T - E)^{\gamma-1} = 0$$

$$\Rightarrow E_T - E = E$$

$$\Rightarrow \boxed{E = E_T/2} \quad \text{as expected}$$

c) The probability for the left side to have energy E is just proportional to the number of states

$$P(E) \propto \Omega(E)\,\Omega(E_T - E) \propto E^{\gamma}(E_T - E)^{\gamma}$$

We can find the width of $P(E)$ by finding the E' such that

$$P(E') = \tfrac{1}{2} P(E_T/2) \quad \text{the most probable value of } E$$

ie we want

$$(E')^{\gamma}(E_T - E')^{\gamma} = \tfrac{1}{2}\left(\tfrac{E_T}{2}\right)^{\gamma}\left(E_T - \tfrac{E_T}{2}\right)^{\gamma}$$

Let $E' = \frac{E_T}{2} + \delta E$ then

$$\left(\frac{E_T}{2} + \delta E\right)^{\gamma}\left(\frac{E_T}{2} - \delta E\right)^{\gamma} = \frac{1}{2}\left(\frac{E_T}{2}\right)^{2\gamma}$$

$$\left(\frac{E_T}{2}\right)^{2\gamma}\left(1 + \frac{\delta E}{(E_T/2)}\right)^{\gamma}\left(1 - \frac{\delta E}{(E_T/2)}\right)^{\gamma} = \frac{1}{2}\left(\frac{E_T}{2}\right)^{2\gamma}$$

$$\left[1 - \frac{\delta E^2}{(E_T/2)^2}\right]^{\gamma} = \frac{1}{2}$$

$$1 - \frac{\delta E^2}{(E_T/2)^2} = \left(\frac{1}{2}\right)^{1/\gamma}$$

$$\frac{\delta E^2}{(E_T/2)^2} = 1 - \left(\frac{1}{2}\right)^{1/\gamma} = 1 - e^{-\frac{1}{\gamma}\ln 2}$$

as N gets large, $1/\gamma$ gets small

$$\frac{\delta E^2}{(E_T/2)^2} = 1 - e^{-\frac{1}{\gamma}\ln 2} \approx 1 - \left(1 - \frac{1}{\gamma}\ln 2\right)$$

$$\approx \frac{1}{\gamma}\ln 2$$

$$\frac{\delta E^2}{(E_T/2)^2} \propto \frac{1}{\gamma} = \frac{2}{3N-2} \sim \frac{1}{N}$$

so width $\boxed{\frac{\delta E}{(E_T/2)} \sim \frac{1}{\sqrt{N}}}$

3)

a) two states for each object $+\epsilon$ or $-\epsilon$

Let N_+ be # objects with $+\epsilon$

N_- be # objects with $-\epsilon$

$$N_+ + N_- = N \quad \text{fixed}$$

$$N_+\epsilon - N_-\epsilon = (N_+ - N_-)\epsilon = E \quad \text{fixed}$$

$$\left.\begin{array}{l} \Rightarrow N_- = N - N_+ \\ (2N_+ - N)\epsilon = E \end{array}\right\} \Rightarrow N_+ = \frac{1}{2}\left(N + \frac{E}{\epsilon}\right) = \frac{N}{2}\left(1 + \frac{E}{N\epsilon}\right)$$

$$N_- = \frac{1}{2}\left(N - \frac{E}{\epsilon}\right) = \frac{N}{2}\left(1 - \frac{E}{N\epsilon}\right)$$

The number of ways to choose N_+ in $+\epsilon$ state and N_- in $-\epsilon$ state, from N total, is

$$\frac{N!}{N_+!\,N_-!}$$

$$\Rightarrow \text{entropy} \quad S = k_B \ln\left(\frac{N!}{N_+!\,N_-!}\right)$$

large N $\rightarrow$ Stirlings approx $\quad \ln N! = N\ln N - N$

$$\frac{S}{k_B} = N\ln N - N - N_+\ln N_+ + N_+ - N_-\ln N_- + N_-$$

$$= N\ln N - N_+\ln N_+ - N_-\ln N_- \quad \text{as } N_+ + N_- = N$$

$$= (N_+ + N_-)\ln N - N_+\ln N_+ - N_-\ln N_-$$

$$\frac{S}{k_B} = N_+ \ln\left(\frac{N}{N_+}\right) + N_- \ln\left(\frac{N}{N_-}\right)$$

this way it is clear S is extensive

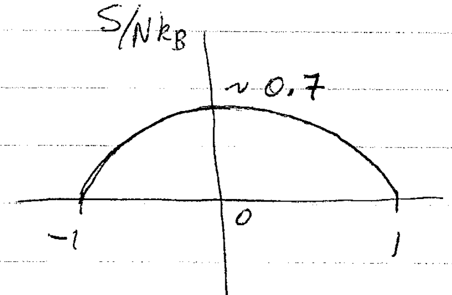
$$\boxed{\frac{S(E,N)}{k_B} = \frac{N}{2}\left(1+\frac{E}{N\epsilon}\right)\ln\left(\frac{2}{1+\frac{E}{N\epsilon}}\right) + \frac{N}{2}\left(1-\frac{E}{N\epsilon}\right)\ln\left(\frac{2}{1-\frac{E}{N\epsilon}}\right)}$$

let $\boxed{x = \frac{E}{N\epsilon}}$

$$\frac{S}{Nk_B} = \left(\frac{1+x}{2}\right)\ln\left(\frac{2}{1+x}\right) + \left(\frac{1-x}{2}\right)\ln\left(\frac{2}{1-x}\right)$$

$$\boxed{\frac{S}{Nk_B} = -\left(\frac{1+x}{2}\right)\ln\left(\frac{1+x}{2}\right) - \left(\frac{1-x}{2}\right)\ln\left(\frac{1-x}{2}\right)}$$

range of x is -1 (all in $-\epsilon$)
to $+1$ (all in $+\epsilon$)



at $x=0$,

$$\frac{S}{Nk_B} = \frac{1}{2}\ln 2 + \frac{1}{2}\ln 2 = \ln 2 = 0.693$$

Unlike a gas where as E increases so $\vec{p}$ increases and there are more states in phase space available to the system, here as E increases above $x=0$ the number of available states decreases! at $x=1$, there is only a single state available!

b) Temperature

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = \frac{1}{N\epsilon}\frac{\partial S}{\partial x}$$

$$\Rightarrow \quad \frac{\epsilon}{k_B T} = \frac{\partial}{\partial x}\left(\frac{S}{Nk_B}\right) = -\frac{1}{2}\ln\left(\frac{1+x}{2}\right) + \frac{1}{2}\ln\left(\frac{1-x}{2}\right)$$

$$-\left(\frac{1+x}{2}\right)\left(\frac{1}{1+x}\right) - \left(\frac{1-x}{2}\right)\left(\frac{-1}{1-x}\right)$$

$$\frac{\epsilon}{k_B T} = \frac{1}{2}\ln\left(\frac{1-x}{1+x}\right)$$

$$\frac{k_B T}{\epsilon} = \frac{2}{\ln\left(\frac{1-x}{1+x}\right)}$$

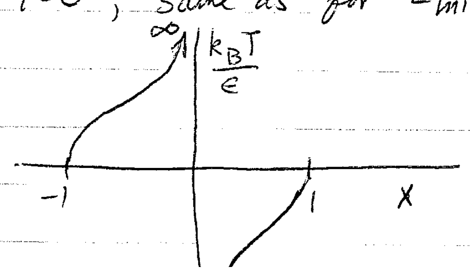
$$\boxed{T = \frac{2\epsilon}{k_B} \frac{1}{\ln\left(\frac{1-E/N\epsilon}{1+E/N\epsilon}\right)}}$$

For $E > 0$, ie $x > 0$ the figure of $\frac{S}{Nk_B}$ vs x clearly shows $\frac{1}{N\epsilon}\frac{\partial S}{\partial x} = \frac{1}{T} < 0$

ie T is negative at $x = 0$, $\frac{1}{N\epsilon}\frac{\partial S}{\partial x} = \frac{1}{T} = 0$, $T \to \infty$

From above, $x > 0 \Rightarrow \ln\left(\frac{1-x}{1+x}\right) < 0$ as $\left(\frac{1-x}{1+x}\right) < 1$
so again we have $T < 0$.

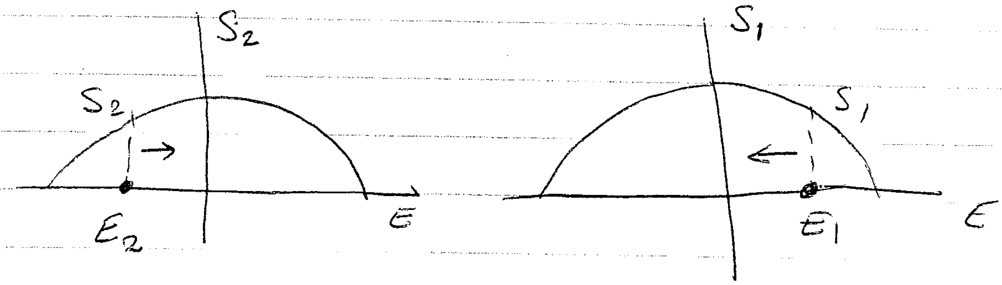
This is reasonable. The state with $E = 0$, ie half up and half down, is the state with the highest degeneracy. As E increases, the degeneracy decreases, so $\frac{\partial S}{\partial E} = \frac{1}{T} < 0$. At $E_{max} = N\epsilon$, or $x = 1$, there is only a single non-degenerate state, all up.
So here $T = 0$, same as for $E_{min} = -N\epsilon$, $x = -1$, all down.



as increase E, $T \to +\infty$ before it goes negative at $-\infty$.

c) system (1) has $T_1 < 0 \Rightarrow E_1 > 0$
system (2) has $T_2 > 0 \Rightarrow E_2 < 0$

$\Rightarrow$ system (2) has less energy than system (1)



we see that if energy flows from system (1) to system (2), ie E_1 decreases and E_2 increases, then both S_2 and S_1 increase. Since the total entropy is maximized, this then is what happens.

$\boxed{\text{heat flows from (1) to (2)}}$

as E_2 increases, T_2 increases
as E_1 decreases, T_1 decreases (becomes more negative)

negative temperature here is "hotter" than positive temperature

system will come into equilibrium when $T_1 = T_2$
subject to constant that $E_1 + E_2$ is kept constant.