Solutions Problem Set 4

1) From the microcanonical ensemble with Gibbs' correction for indistinguishable particles we have the microcanonical partition function

$$\Omega(E) = e^{S(E)/k_B} = \left[\frac{V}{h^3} (2\pi m E)^{3/2} \right]^N \frac{1}{\left(\frac{3N}{2}-1\right)!\, N!} \frac{\Delta E}{E}$$

The canonical partition function is

$$Q_N(T) = \int_0^\infty \frac{dE}{\Delta E} \Omega(E) e^{-\beta E}$$

$$= \left[\frac{V}{h^3} (2\pi m)^{3/2} \right]^N \frac{1}{\left(\frac{3N}{2}-1\right)!\, N!} \int_0^\infty dE \; E^{\frac{3N}{2}-1} \; e^{-\beta E}$$

① we can evaluate the integral exactly by successive integration by parts

$$\int_0^\infty dE \; E^\gamma e^{-\beta E} = \int_0^\infty dE \; \gamma E^{\gamma-1} \frac{e^{-\beta E}}{\beta} \qquad \text{boundary terms vanish}$$

$$= \int_0^\infty dE \; \frac{\gamma(\gamma-1)}{\beta^2} e^{-\beta E}$$

$$= \frac{\gamma!}{\beta^\gamma} \int_0^\infty dE \, e^{-\beta E} = \frac{\gamma!}{\beta^{\gamma+1}}$$

So

$$Q_N(T) = \left[\frac{V}{h^3} (2\pi m)^{3/2} \right]^N \frac{1}{\left(\frac{3N}{2}-1\right)!\, N!} \left(\frac{3N}{2}-1\right)! \, (k_B T)^{\frac{3N}{2}}$$

$$\boxed{Q_N(T) = \left[\frac{V}{h^3} (2\pi m k_B T)^{3/2} \right]^N \frac{1}{N!}}$$

The Helmholtz free energy is then

$$A(T,V,N) = -k_BT \ln Q_N(T,V)$$

$$= -k_BTN \ln\left[\frac{V}{h^3}(2\pi m k_BT)^{3/2}\right] + k_BT \ln N!$$

$$= -k_BTN \ln\left[\frac{V}{h^3}(2\pi m k_BT)^{3/2}\right] + \underbrace{k_BT N \ln N - k_BTN}_{\text{using Stirling's formula}}$$

$$\boxed{A(T,V,N) = -k_BTN\left[1 + \ln\left[\frac{V}{Nh^3}(2\pi m k_BT)^{3/2}\right]\right]}$$

From problem set 1 we had

$$A(T,V,N) = \left(\frac{3}{2}k_B - \frac{S_0}{N_0}\right)NT - Nk_BT \ln\left[\frac{V}{V_0}\frac{N_0}{N}\left(\frac{3N_0k_BT}{2E_0}\right)^{3/2}\right]$$

If we identify $S_0 = \frac{5}{2}N_0k_B$ and $\frac{N_0}{V_0}\left(\frac{3}{2}\frac{N_0}{E_0}\right)^{3/2} = \frac{(2\pi m)^{3/2}}{h^3}$

Then the two expressions for A become the same. In particular, the dependencies of A on N, V, T are the same in both expressions

To finish we note that $\frac{N_0}{V_0}\left(\frac{3}{2}\frac{N_0}{E_0}\right)^{3/2}$ is <u>intensive</u> just as is $\frac{(2\pi m)^{3/2}}{h^3}$, and they both have the same physical dimensions of $\frac{(\text{time})^3}{(\text{mass})^{3/2}(\text{length})^6}$

2)

One dimensional particles - non-interacting

$$H = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + U(x_i)\right]$$

p_i is momentum of i
x_i is position of i

$$U(x) = \begin{cases} 0 & 0 \le x < \frac{L}{2} \\ U_0 & \frac{L}{2} \le x \le L \end{cases}$$

The density matrix is (see Notes Eq. (2.8.13))

$$\rho(x_1 \ldots x_N, p_1 \ldots p_N) = \frac{e^{-\beta H(x_1 \ldots x_N, p_1 \ldots p_N)}}{\int dx_1 \ldots dx_N \, dp_1 \ldots dp_N \, e^{-\beta H(x_1 \ldots x_N, p_1 \ldots p_N)}}$$

To find the probability particle 1 is at x_1, we integrate ρ over all the p_i and over x_i $i = 2, \ldots N$

Since particles are non-interacting, the Boltzmann factor can be factored

$$e^{-\beta H} = e^{-\beta H_1} e^{-\beta H_2} \ldots e^{-\beta H_N}$$

where $H_i = \frac{p_i^2}{2m} + U(x_i)$ is the single particle Hamiltonian for particle i

$$P(x_1) = \frac{\int dx_2 \cdots dx_N \int dp_1 \cdots dp_N \, e^{-\beta H_1} e^{-\beta H_2} \cdots e^{-\beta H_N}}{\int dx_1 \cdots dx_N \int dp_1 \cdots dp_N \, e^{-\beta H_1} e^{-\beta H_2} \cdots e^{-\beta H_N}}$$

$$= \frac{\int dp_1 \, e^{-\beta H_1} \left(\int dx_2 dp_2 \, e^{-\beta H_2} \right) \cdots \left(\int dx_N dp_N \, e^{-\beta H_N} \right)}{\left(\int dx_1 dp_1 \, e^{-\beta H_1} \right) \left(\int dx_2 dp_2 \, e^{-\beta H_2} \right) \cdots \left(\int dx_N dp_N \, e^{-\beta H_N} \right)}$$

all factors for particles $i = 2 \ldots N$ in numerator cancel corresponding factors in denominator

$$P(x_1) = \frac{\int dp_1 \, e^{-\beta p_1^2/2m} \, e^{-\beta U(x_1)}}{\int dx_1 dp_1 \, e^{-\beta p_1^2/2m} \, e^{-\beta U(x_1)}}$$

integral over p_1 in numerator cancels that in denominator

$$P(x_1) = \frac{e^{-\beta U(x_1)}}{\int dx_1 \, e^{-\beta U(x_1)}} = \frac{e^{-\beta U(x_1)}}{\frac{L}{2}\left(1 + e^{-\beta U_0}\right)}$$

$$P(x_1) = \begin{cases} \dfrac{2}{L(1+e^{-\beta U_0})} & 0 \leq x_1 < \frac{L}{2} \\[2ex] \dfrac{2e^{-\beta U_0}}{L(1+e^{-\beta U_0})} & \frac{L}{2} \leq x_1 \leq L \end{cases}$$

Probability the particle in right half of box is

$$p = \int_{\frac{L}{2}}^{L} dx_1\, P(x_1) = \frac{L}{2}\,\frac{2e^{-\beta u_0}}{L(1+e^{-\beta u_0})}$$

$$\boxed{\begin{aligned} p &= \frac{e^{-\beta u_0}}{1+e^{-\beta u_0}} \quad \text{prob particle in right half} \\ q = 1-p &= \frac{1}{1+e^{-\beta u_0}} \quad \text{prob particle in left half} \end{aligned}}$$

The probability that M of the N particles are in the right half of the box is given by the binomial distribution

$$\boxed{P(M) = \frac{N!}{M!\,(N-M)!}\, p^{M}\, q^{N-M}}$$

The average number in right half is $\langle M \rangle = Np$

The standard deviation of the number in the right half is

$$\sigma_M = \sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{Npq}$$

(see the end of Notes 2-15 if this is not familiar to you)

We understand the binomial distribution as follows.

Suppose particles were distinguishable. Then the probability to get a particular set of M particles on the right side would be $p^M q^{N-M}$

But for P(M) we do not care about which particles are on the right side, so we must multiply $p^M q^{N-M}$ by the number of ways we can choose M of the particles to be on the right. This is the combinatoric factor $N!/M!(N-M)!$

We have N ways to choose the first particle that is on the right, N−1 ways to choose the 2nd, ..., N−(M−1) ways to choose the last, M^{th}, particle. This is then

$$N(N-1)(N-2)\cdots(N-M+1) = \frac{N!}{(N-M)!} \text{ ways}$$

But now we do not care what is the order in which we picked the M particles to put on the right side. So how many ways are there to order those M? There are

$M(M-1)(M-2)\cdots(1) = M!$ ways.

So the number of ways to put M particles out of N on the right hand side, where we do not care about the order in which the particles are added to the right side are

$$\frac{N!}{M!(N-M)!}$$

In our discussion session the question was raised whether this factor should be different because the particles are <u>indistinguishable</u>. I think the answer is <u>NO</u>.

Even if particles are <u>indistinguishable</u>, there are still N of them. Imagine putting the N particles in a bag, reaching in and picking out M of them to put in the right side of the box. There are

$\frac{N!}{M!(N-M)!}$ ways to do that - it does not matter whether we look at the chosen particles (so as to try and distinguish which they are) or whether we don't look (because we know that they can't be distinguished). It is still the same factor $\frac{N!}{M!(N-M)!}$.

If particles were *distinguishable*, we could ask the question – what is the probability that particles 1, 3, 6, 47, 96, ... are the M particular particles that are on the right side. That probability would be $p^M q^{N-M}$.

But if the particles are *indistinguishable* we cannot ask that question because we cannot say which particle is which. But the probability that there are M particles on the right side is independent of whether they are *distinguishable* or *indistinguishable*!

It would *not* be correct to think that when you reach into the bag to select the first particle to put in the right side of the box that there is only one way to do that since all particles are identical. The particles can all be in different states characterized by different values of p and x, so there are still M choices.

3)

a) Canonical ensemble

$$Q_N(T) = \sum_i e^{-\beta E_i}$$ E_i is total energy of state i

Since the degrees of freedom do not interact, we have

$$Q_N = Q_1^N$$ (no $\frac{1}{N!}$ since distinguishable)

where Q_1 is the one object partition function

$$Q_1 = e^{-\beta \epsilon} + e^{\beta \epsilon}$$ (two states with energies $+\epsilon$ and $-\epsilon$)

$$Q_N = \left(e^{\beta\epsilon} + e^{-\beta\epsilon}\right)^N = \left[2\cosh(\beta\epsilon)\right]^N$$

Helmholtz free energy $A(T,N)$

$$\boxed{A = -k_B T \ln Q_N = -N k_B T \ln\left(e^{\beta\epsilon} + e^{-\beta\epsilon}\right)}$$

b) entropy from canonical distribution

$$S = -\frac{\partial A}{\partial T} = N k_B \ln\left(e^{\beta\epsilon} + e^{-\beta\epsilon}\right) + N k_B T \frac{\left(-\frac{\epsilon}{k_B T^2} e^{\beta\epsilon} + \frac{\epsilon}{k_B T^2} e^{-\beta\epsilon}\right)}{e^{\beta\epsilon} + e^{-\beta\epsilon}}$$

$$S = N k_B \ln\left(e^{\beta\epsilon} + e^{-\beta\epsilon}\right) - \frac{N\epsilon}{T} \frac{e^{\beta\epsilon} - e^{-\beta\epsilon}}{e^{\beta\epsilon} + e^{-\beta\epsilon}}$$

we need to find a relation for T in terms of E and substitute it into the above

$$E = -\frac{\partial}{\partial\beta}(\ln Q_N) = \frac{\partial}{\partial\beta}\left(\frac{A}{k_B T}\right)$$

$$= \frac{\partial}{\partial\beta}\left[-N\ln(e^{\beta\epsilon}+e^{-\beta\epsilon})\right]$$

$$E = -N\epsilon\,\frac{e^{\beta\epsilon}-e^{-\beta\epsilon}}{e^{\beta\epsilon}+e^{-\beta\epsilon}} = -N\epsilon\tanh\beta\epsilon$$

need to solve for T in terms of E and substitute in expression for S.

let $y = e^{\beta\epsilon}$ then $-\frac{E}{N\epsilon} = -x = \frac{y-1/y}{y+1/y} = \frac{y^2-1}{y^2+1}$

$x = \frac{E}{N\epsilon}$

$$\Rightarrow -xy^2 - x = y^2 - 1 \Rightarrow y^2 = \frac{1-x}{1+x} \Rightarrow y = e^{\beta\epsilon} = \sqrt{\frac{1-x}{1+x}}$$

substitute into S to get

$$\frac{S}{Nk_B} = \ln\left(y+\frac{1}{y}\right) - (\ln y)(-x)$$

$$= \ln\left(\sqrt{\frac{1-x}{1+x}}+\sqrt{\frac{1+x}{1-x}}\right) + x\ln\left(\frac{1-x}{1+x}\right)^{1/2}$$

$$= \ln\left(\frac{(1-x)+(1+x)}{\sqrt{(1-x)(1+x)}}\right) + \frac{x}{2}\ln\left(\frac{1-x}{1+x}\right)$$

$$= \ln 2 - \frac{1}{2}\ln(1-x) - \frac{1}{2}\ln(1+x) + \frac{x}{2}\ln(1-x)$$
$$- \frac{x}{2}\ln(1+x)$$

$$\frac{S}{Nk_B} = \ln 2 - \frac{1}{2}(1-x)\ln(1-x) - \frac{1}{2}(1+x)\ln(1+x)$$

write $\ln 2 = -\frac{1}{2}(1-x)\ln\left(\frac{1}{2}\right) - \frac{1}{2}(1+x)\ln\left(\frac{1}{2}\right)$
to get

$$\boxed{\frac{S}{Nk_B} = -\frac{1}{2}(1-x)\ln\left(\frac{1-x}{2}\right) - \frac{1}{2}(1+x)\ln\left(\frac{1+x}{2}\right)}$$

exact same result as in part (a) of problem (1)

Hence we get same results whether we use the microcanonical or the canonical ensemble.

A further discussion of Problem Set 4, problem 2

When we compute the probability $P(M)$ that M of the N particles are found on the right hand side of the box, does it matter if the particles are *distinguishable* or *indistinguishable*? We consider both cases explicitly, and conclude that both cases give the same result for $P(M)$.

Distinguishable particles

The probability density ρ^{dis} for the system of distinguishable particles to have the N particles at coordinates $\{x_i\}$ with momena $\{p_i\}$ is,

$$\rho^{\text{dis}}(\{x_i,p_i\}) = \frac{\mathrm{e}^{-\beta\mathcal{H}(\{x_i,p_i\})}}{\left(\prod_i \int dx_i dp_i\right)\mathrm{e}^{-\beta\mathcal{H}(\{x_i,p_i\})}} \tag{1}$$

where ρ^{dis} is normalized so that

$$\int dx_1 dp_1 \cdots dx_N dp_N \, \rho^{\text{dis}}(x_1,p_1,\ldots,x_N,p_N) = 1 \tag{2}$$

Since the particles are non-interacting, $\mathcal{H}(\{x_i,p_i\}) = \sum_{i=1}^{N} \mathcal{H}^{(1)}(x_i,p_i)$, and this becomes,

$$\rho^{\text{dis}}(\{x_i,p_i\}) = \frac{\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_1,p_1)}\cdots\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_N,p_N)}}{\left(\int dx_1 dp_1\,\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_1,p_1)}\right)\cdots\left(\int dx_N dp_N\,\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_N,p_N)}\right)} \tag{3}$$

$$= \left(\frac{\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_1,p_1)}}{\int dx_1 dp_1\,\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_1,p_1)}}\right)\cdots\left(\frac{\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_N,p_N)}}{\int dx_N dp_N\,\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x_N,p_N)}}\right) \tag{4}$$

$$= \rho_1(x_1,p_1)\cdots\rho_1(x_N,p_N) \qquad \text{where} \qquad \rho_1(x,p) \equiv \frac{\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x,p)}}{\int dx dp\,\mathrm{e}^{-\beta\mathcal{H}^{(1)}(x,p)}} \tag{5}$$

Since the particles are non-interacting, they are statistically independent, so the joint N-particle probability density $\rho^{\text{dis}}(\{x_i,p_i\})$ factors into a product of N single-particle probability densities $\rho_1(x,p)$. That is always true of independent random variables – the joint probability distribution factors into a product of distributions for the individual random variables.

Now we are interested only in the probability for the position, so we integrate over the momentum. Since $\mathcal{H}^{(1)} = \dfrac{p^2}{2m} + U(x)$ we have

$$\rho_1(x) = \int dp\,\rho_1(x,p) = \frac{\mathrm{e}^{-\beta U(x)}\int dp\,\mathrm{e}^{-\beta p^2/2m}}{\int dx\,\mathrm{e}^{-\beta U(x)}\int dp\,\mathrm{e}^{-\beta p^2/2m}} = \frac{\mathrm{e}^{-\beta U(x)}}{\int dx\,\mathrm{e}^{-\beta U(x)}} \tag{6}$$

For $U(x) = \begin{cases} 0 & 0 \le x < L/2 \\ U_0 & L/2 \le x \le L \end{cases}$ we have $\int dx\, \mathrm{e}^{-\beta U(x)} = \frac{L}{2}\left[1 + \mathrm{e}^{\beta U_0}\right]$, so

$$\boxed{\rho_1(x) = \frac{2\,\mathrm{e}^{-\beta U(x)}}{L\left[1 + \mathrm{e}^{-\beta U_0}\right]}} \tag{7}$$

The probability the particle will be found in the right hand side of the box is then,

$$p = \int_{L/2}^{L} dx\, \rho_1(x) = \frac{L}{2}\,\frac{2\mathrm{e}^{-\beta U_0}}{L\left[1 + \mathrm{e}^{-\beta U_0}\right]} = \boxed{\frac{\mathrm{e}^{-\beta U_0}}{\left[1 + \mathrm{e}^{-\beta U_0}\right]} = p} \tag{8}$$

and the probability the particle will be found in the left hand side of the box is,

$$\boxed{q = 1 - p = \frac{1}{\left[1 + \mathrm{e}^{-\beta U_0}\right]}} \tag{9}$$

Back now to the N-particle system, the probability that we have particles i at positions x_i is given by,

$$\rho^{\mathrm{dis}}(x_1, \ldots, x_N) = \rho_1(x_1) \cdots \rho_1(x_N) \qquad \text{since we just integrate Eq. (5) over all the } p_i \tag{10}$$

The probability that we will have the specific particles $i = 1, \ldots, M$ on the right side, and $i = M+1, \ldots, N$ on the left side, is then obtained by integrating each of the $\rho_1(x)$ over the appropriate interval. We get,

$$P = p^M q^{N-M} \tag{11}$$

But if we want to know the probability that M of the particles are on the right side, and all the others are on the left side, and we don't care which are the ones that are on the right, then that probability is,

$$\boxed{P(M) = \frac{N!}{M!(N-M)!}\, p^M q^{N-M}} \tag{12}$$

since there are $\frac{N!}{M!(N-M)!}$ ways to choose which M of the N particles to put on the right side.

Indistinguishable particles

Now suppose our particles are non-interacting but are indistinguishable. Now the N-particle probability density ρ^{indis} should be normalized so,

$$\frac{1}{N!}\int dx_1 dp_1 \cdots dx_N dp_N\, \rho^{\mathrm{indis}}(x_1, p_1, \ldots, x_N, p_N) = 1 \tag{13}$$

The $1/N!$ is there because we do not want to over-count states, i.e. the configuration $(x_1, p_1, x_2, p_2, \ldots, x_N, p_N)$ is the same as the configuration $(x_2, p_2, x_1, p_1, \ldots, x_N, p_N)$. So $\rho(x_1, p_1, \ldots, x_N, p_N)$ is the probability density that one particle has coordinates (x_1, p_1), another has coordinates (x_2, p_2), and so on, and we don't care which particle has which coordinates because they are indistinguishable.

Comparing to Eq. (2) we can therefore write,

$$\rho^{\mathrm{indis}}(\{x_i, p_i\}) = N!\, \rho^{\mathrm{dis}}(\{x_i, p_i\}) \tag{14}$$

And similarly, integrating over the momenta, the joint probability to find one particle at x_1, another at x_2, and so on, is,

$$\rho^{\text{indis}}(x_1, x_2, \ldots, x_N) = N!\, \rho^{\text{dis}}(x_1, x_2, \ldots, x_N) = N!\, \rho_1(x_1) \cdots \rho_1(x_N) \tag{15}$$

Now suppose I have M red particles and $N - M$ blue particles in the box. The red particles are indistinguishable from each other, and the blue particles are indistinguishable from each other, but the red particles can be distinguished from the blue particles. The probability that the red particles are at $(x_1, \ldots, x_M)$ and the blue particles are at (x_{M+1}, x_N) would be,

$$\rho^{\text{indis}}(x_1, \ldots, x_M)\, \rho^{\text{indis}}(x_{M+1}, \ldots, x_N) = \Big[M!\, \rho(x_1) \cdots \rho(x_M)\Big] \Big[(N-M)!\, \rho(x_{M+1}) \cdots \rho(x_N)\Big] \tag{16}$$

$$= M!(N-M)!\, \rho_1(x_1) \cdots \rho_1(x_N) \tag{17}$$

Comparing to Eq. (15) we therefore have,

$$\boxed{\rho^{\text{indis}}(x_1, x_2, \ldots, x_N) = \frac{N!}{M!(N-M)!}\, \rho^{\text{indis}}(x_1, \ldots, x_M)\, \rho^{\text{indis}}(x_{M+1}, \ldots, x_N)} \tag{18}$$

If we recall that in the *microcanonical* ensemble, the probability to be in a particular state is $1/\Omega$, then the above is similar to Eq. (2.7.25) in our discussion of the entropy of mixing.

So, using Eq. (18), the probability $P(M)$ that M of the indistinguishable particles are on the right and $N - M$ are on the left is,

$$P(M) = \int\limits_{\substack{\text{such that} \\ M \text{ of the } x_i \text{ have } L/2 \leq x_i \\ N-M \text{ of the } x_i \text{ have } x_i < L/2 \\ \text{without double counting configurations}}} dx_1 \cdots dx_N\, \rho^{\text{indis}}(x_1, \cdots, x_N) \tag{19}$$

$$= \frac{N!}{M!(N-M)!} \int\limits_{\substack{\text{such that} \\ \text{all } M \text{ of the } x_i \text{ have } L/2 \leq x_i \\ \text{without double counting configurations}}} dx_1 \cdots dx_M\, \rho^{\text{indis}}(x_1, \cdots, x_M) \tag{20}$$

$$\times \int\limits_{\substack{\text{such that} \\ \text{all } N-M \text{ of the } x_i \text{ have } x_i < L/2 \\ \text{without double counting configurations}}} dx_{M+1} \cdots dx_N\, \rho^{\text{indis}}(x_{M+1}, \cdots, x_N)$$

Now we have for the first term on the rightmost side of the above equation,

$$P_R = \int\limits_{\substack{\text{such that} \\ \text{all } M \text{ of the } x_i \text{ have } L/2 \leq x_i \\ \text{without double counting configurations}}} dx_1 \cdots dx_M\, \rho^{\text{indis}}(x_1, \cdots, x_M) \tag{21}$$

$$= \frac{1}{M!} \int_{L/2}^{L} dx_1 \cdots dx_M\, \rho^{\text{indis}}(x_1, \ldots, x_M) = \frac{1}{M!} \int_{L/2}^{L} dx_1 \cdots dx_M \Big[M!\, \rho^{\text{dis}}(x_1, \ldots, x_M)\Big] \tag{22}$$

$$= \int_{L/2}^{L} dx_1 \cdots dx_M\, \rho_1(x_1) \cdots \rho_1(x_M) = p^M \tag{23}$$

while the second term is,

$$P_L \quad = \int\limits_{\substack{\text{such that} \\ \text{all } N-M \text{ of the } x_i \text{ have } x_i < L/2 \\ \text{without double counting configurations}}} dx_{M+1} \cdots dx_N \, \rho^{\text{indis}}(x_{M+1}, \cdots, x_N) \tag{24}$$

$$= \frac{1}{(N-M)!} \int_0^{L/2} dx_{M+1} \cdots dx_N \, \rho^{\text{indis}}(x_{M+1}, \ldots, x_N) \tag{25}$$

$$= \frac{1}{(N-M)!} \int_0^{L/2} dx_{M+1} \cdots dx_N \left[(N-M)! \, \rho^{\text{dis}}(x_{M+1}, \ldots, x_N) \right] \tag{26}$$

$$= \int_0^{L/2} dx_{M+1} \cdots dx_N \, \rho_1(x_{M+1}) \ldots \rho_1(x_N) = q^{N-M} \tag{27}$$

Putting these results into Eq. (20) we get,

$$P(M) = \frac{N!}{M!(N-M)!} P_R \, P_L = \frac{N!}{M!(N-M)!} \, p^N \, q^{N-M} \tag{28}$$

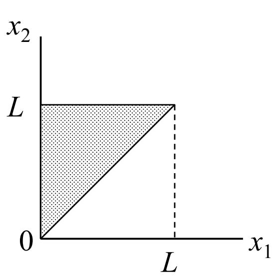
This is exactly the same answer that we had for distinguishable particles!

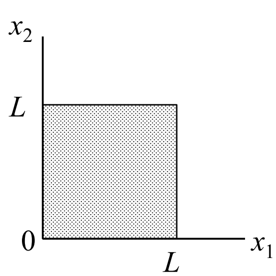
In several places above we discussed doing integrals without double counting states for identical particles. To be specific about what we mean, suppose the coordinates of the N particles are (x_1, p_1), (x_2, p_2), $\ldots (x_N, p_N)$. Then if we want to integrate without double counting, we should integrate the normalization condition as,

$$\int_{-\infty}^{\infty} dp_N \cdots \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_1 \int_{x_{N-1}}^{L} dx_N \cdots \int_{x_1}^{L} dx_2 \int_0^L dx_1 \, \rho^{\text{indis}}(x_1, p_1, x_2, p_2, \ldots, x_N, p_N) = 1 \tag{29}$$

That is, we first choose $x_1 \in [0, L]$, then we should next choose $x_2 \in [x_1, L]$, then $x_3 \in [x_2, L]$, etc., so that the position coordinates are ordered as $0 \leq x_1 \leq x_2 \leq \cdots \leq x_N \leq L$. This way if (x_1, x_2) is in the region of integration, then (x_2, x_1) is not, and so we do not double count. Alternatively, we could integrate over $x_i \in [0, L]$ for all x_i, but then we need to divide the integration by the factor $N!$ because we are double counting.

To see this graphically, consider the case of just two particles. By the above, we want to integrate over $x_1 \in [0, L]$ and $x_2 \in [x_1, L]$. Graphically this is the shaded region shown below to the left. Alternatively, we could integrate over $x_1 \in [0, L]$ *and* $x_2 \in [0, L]$, shown as the shaded region below to the right. But this region has twice the area as the one to the left, so we would have to multiply by $1/2 = 1/2!$ to get the same answer as when we integrate over the region to the left.





If we had distinguishable particles, then (x_1, x_2) is a different state from (x_2, x_1) and we would integrate over the region above to the right.

One then has (imagine we have already integrated over the p_i),

$$\frac{1}{2!} \int_0^L dx_2 \int_0^L dx_1 \, \rho^{\text{indis}}(x_1, x_2) = \int_{x_1}^{L} dx_2 \int_0^L dx_1 \, \rho^{\text{indis}}(x_1, x_2) = 1 \tag{30}$$

while

$$\int_0^L dx_2 \int_0^L dx_1 \, \rho^{\mathrm{dis}}(x_1, x_2) = 1 \tag{31}$$

This leads to

$$\frac{1}{2!}\rho^{\mathrm{indis}}(x_1, x_2) = \rho^{\mathrm{dis}}(x_1, x_2) \quad \Rightarrow \quad \rho^{\mathrm{indis}}(x_1, x_2) = 2!\, \rho^{\mathrm{dis}}(x_1, x_2) \tag{32}$$

ρ^{indis} must be twice as large as ρ^{dis} because when we normalize we are really integrating ρ^{indis} over only halve the area as when we integrate ρ^{dis}.

For N particles, this generalizes to $\rho^{\mathrm{indis}}(x_1, \ldots, x_N) = N!\rho^{\mathrm{dis}}(x_1, \ldots, x_N)$.