Solutions Problem Set 9 PHYS 418 Sitefon - Boltzam law: block body enutts I) power per unit surface aca => Total power emitted by sin is P = 0 T_8 4TT R_5 where Ts is temp of sun ad Rs is radiu of sun The fraction of this power that hits, ad gets absorbed by the earth is the fraction of the solid angle that the earth subferreds. Re= raduis of earth sun Ro= radus of earth's orbit about the sun solid angle $Q = \frac{\pi Re^2}{4\pi} = \frac{area earth presents to sun}{area of sphere of radio Ro$ power absorbed by earth & therefore $\left(\sigma T_{s}^{4} 4 \pi R_{s}^{2}\right) \left(\frac{\pi R_{e}^{2}}{4 \pi R^{2}}\right) = \sigma T_{s}^{4} \pi R_{s}^{2} \frac{R_{e}^{2}}{\pi^{2}}$ The power absorbed by earth should equal the power the earth emitts as a black body $= \sigma T_5^4 \pi R_5^2 R_e^2 = \sigma T_e^4 4 \pi R_e^2 T_e = teup earth$ $\left(\frac{T_e}{T_s}\right)^4 = \left(\frac{R_s}{2R_s}\right)^2 \qquad = \int \frac{T_e}{T_s} \int \frac{R_s}{2R_s} \sim 300^\circ K$

a) Density of states. We start by finding the number of States per unit volume with energy less than or equal to E. Such states lie in k-space within a circle of radius $k = \sqrt{2m\epsilon/k^2}$ $G(\varepsilon) = \frac{1}{1^2} \frac{g_s \pi k^{\perp}}{(\Delta k)^2} = \frac{1}{k^2} \frac{g_s \pi 2m\varepsilon}{(\frac{2\pi}{L})^2 \pi^2} = \frac{g_s m\varepsilon}{2\pi \hbar^2}$ where g_{s} is the spin degeneracy factor The density of states is then $g(\varepsilon) = \frac{dG}{d\varepsilon} = \frac{g_{s}m}{2\pi \hbar^{2}} = \frac{g_{o}}{a \cosh t}$ milep of ε 5) Fermi energy: at T=0, the density is given by $m = \int_{0}^{2F} d\varepsilon g(\varepsilon) = g_{0} \varepsilon_{F} \implies \overline{g}_{0} \varepsilon_{F} \implies \overline{g}_{0} \varepsilon_{F} = \frac{2\pi \hbar^{2} m}{g_{0}}$ The energy density is $\mathcal{U}_{b} = \int_{0}^{\mathcal{E}_{F}} d\mathcal{E} g(\mathcal{E}) \mathcal{E} = \frac{1}{2} g_{0} \mathcal{E}_{F}^{2} = \frac{1}{2} \frac{m^{2}}{\int_{0}^{\infty} -\frac{\pi h^{2} m^{2}}{g_{s} m}}$ $\int u_0 = \frac{1}{2} g \varepsilon_F^2 = \frac{1}{2} \frac{\pi^2}{g_0} = \frac{\pi t^2 \pi^2}{g_{\rm s} m}$ c) at funte T $M = \int_{\partial}^{\partial} d\varepsilon \frac{q(\varepsilon)}{\beta(\varepsilon-\mu)} = g_{\partial} \int_{\partial}^{\partial} d\varepsilon \frac{1}{\varepsilon^{-1}e^{\beta\varepsilon}+1}$

ect x = BE $= g_{o} k_{B} T \int_{0}^{\infty} dx \frac{1}{z^{2} e^{x} + 1}$ look up in integral hardbook = $g_{okgT}\left(l_{m}\left[\frac{l}{e^{-x}+z^{-1}}\right]\right)$ = gokgT (In Z - In 1+Z-1) = JokoT (ln Z - ln Z/+Z) = JokoT ln (1+Z) $M = g_{0}k_{0}T \ln (1+e^{\beta\mu})$ as $Z = e^{\beta\mu}$ From patrice Solve for p Ite = e m/goket $\mu = k_{\rm B} T \ln \left[e^{n/g_{\rm c} k_{\rm B} T} - 1 \right]$ from (b) we can use $\frac{M}{40} = \epsilon_F$ to set $\mu = k_{\rm B}T \ln \left[e^{\epsilon_{\rm F}/k_{\rm B}T} - 1 \right]$ = $k_B T lm \left[e^{\epsilon F/k_B T} \left(1 - e^{-\epsilon T/k_B T} \right) \right]$ $M = E_F + k_B T - \ln \left(1 - e^{-E_F/k_B T} \right)$ mote: as T>0, M=6F, as it should !

 $\mu = \epsilon_{F} \left[1 + \left(\frac{k_{B}T}{\epsilon_{F}} \right) l_{W} \left(1 - e^{-\epsilon_{F}/k_{B}T} \right) \right]$ for small T, $k_BT \ll E_F$, then $e^{-G_F/k_BT} \ll 1$ we can expand $-\ln(1-5) \simeq -8$ small s $\mu \simeq \epsilon_{\rm F} \left[1 - \left(\frac{k_{\rm B}T}{\epsilon_{\rm F}} \right) e^{-\epsilon_{\rm F}/k_{\rm B}T} \right]$ Note that e == [kot has an essented singulaty at T=0 => µ(T) Los not have any Taylor series expansion in T! This is infine the situation in 3D othere M(T) can be expressed as an expansion in (T/TF).

The 2D Fermi Gas and the Sommerfeld Expansion

In Problem Set 9, problem 2, you found that for the 2D fermi gas the density of states is a constant $g(\epsilon) = g_0 = m/\pi\hbar^2$, and that the chemical potential varied with temperature as,

$$\mu(T) = \epsilon_F + k_B T \ln\left[1 - e^{-\epsilon_F/k_B T}\right] \approx \epsilon_F \left[1 - \frac{k_B T}{\epsilon_F} e^{-\epsilon_F/k_B T}\right] \qquad \text{when } k_B T \ll \epsilon_F \tag{1}$$

 $\mu(T)$ thus has an essential singularity at T = 0, which means that a Taylor series for μ , expanding about T = 0, has a zero radius of convergence.

That is consistent with what one find from the Sommerfeld expansion, where to lowest order we have from Eq. (3.9.65),

$$\mu(T) = \epsilon_F \left[1 - \frac{(\pi k_B T)^2}{6} \frac{\epsilon_F}{g(\epsilon_F)} g'(\epsilon_F) \right]$$
(2)

Since $g(\epsilon) = g_0$ a constant, $g'(\epsilon) = 0$, and so the Sommerfeld expansion gives $\mu(T) = \epsilon_F$. Higher powers in the Sommerfeld expansion also vanish since all derivatives of $g(\epsilon)$ vanish. The Sommerfeld expansion therefore concludes $\mu(T) = \epsilon_F$ stays constant at all T. The Sommerfeld expansion cannot capture the essential singularity at T = 0, because the essential singularity has no convergent Taylor series expansion!

Yet the Sommerfeld expansion is still not doing badly! The relative correction term from the T = 0 value goes as,

$$\frac{\epsilon_F - \mu(T)}{\epsilon_F} \approx \frac{T}{T_F} \,\mathrm{e}^{-T_F/T} \ll 1 \qquad \text{since } T_F = \epsilon_F/k_B \approx 5000 \,\,\mathrm{K} \tag{3}$$

So at room temperature, $T_F/T \approx 17$, and the relative correction is $\approx 2.5 \times 10^{-9}$, and so negligible!

We can see where the essential singularity in $\mu(T)$ comes from as follows. The areal density of particles n is given by,

$$n = \int_0^\infty d\epsilon \, g(\epsilon) f(\epsilon - \mu) = \int_0^\infty d\epsilon \, \frac{g(\epsilon)}{\mathrm{e}^{\beta(\epsilon - \mu)} + 1} = g_0 \int_{-\mu}^\infty dx \, \frac{1}{\mathrm{e}^{\beta x} + 1} \qquad \text{with } x = \epsilon - \mu \tag{4}$$

The density n is fixed and so must be independent of the temperature T. One chooses $\mu(T)$ to give the fixed constant value of n.



To the left we sketch $f(x) = 1/(e^{\beta x} + 1)$ vs x for T = 0and for a finite T > 0. The density n is just proportional to the area under the curve. Because df/dx is symmetric about x = 0, the area under the T = 0 and the T > 0 curves would be equal, and hence μ would be the same, *IF* the range of integration was symmetric about x = 0.

But the range of integration is not symmetric. For x < 0 the integration cuts off at $x = -\mu$, while for x > 0 the integration extends to infinity. For this reason, if μ is kept constant, the area under the T > 0 curve is ever so slightly larger than the area under the T = 0 curve. Thus, to keep n constant, it is necessary that $\mu(T)$ must be slightly smaller than $\mu(0) = \epsilon_F$.

So the shift in μ at finite T has to do with the fact that our integral on x starts at $-\mu$, rather than $-\infty$. The difference in area between the T = 0 curve and the finite T > 0 curve, due to the integral being cut off at $-\mu$ rather than $-\infty$, is then,

$$\int_{-\infty}^{-\mu} dx \, [1 - f(x)] = \int_{-\infty}^{-\mu} dx \, \left[1 - \frac{1}{e^{\beta x} + 1} \right] = \int_{-\infty}^{-\mu} dx \, \frac{e^{\beta x}}{e^{\beta x} + 1} \approx \int_{-\infty}^{-\mu} dx \, e^{\beta x} = k_B T \, e^{-\mu/k_B T} \tag{5}$$

since $e^{\beta x}$ is very small when x < 0 and $\mu/k_B T \gg 1$. This, therefore gives rise to the essential singularity in $\mu(T)$.

But this sort of effect is inherent throughout the Sommerfeld expansion. It arises in the step going from Eq. (3.9.46) to (3.9.47), where we replaced the lower limit of the integrals by $x \to -\infty$. Usually we do not worry about this approximation since it gives such a small correction when the terms in the Sommerfeld expansion are finite. But for the 2D fermi gas, when we compute the density n, the terms in the Sommerfeld expansion all vanish, and so this usually negligible effect is all there is to give a finite difference between the T > 0 result and the T = 0 result. The same would be true for any situation where the quantity one is averaging over $X(\epsilon)$ is such that $g(\epsilon)X(\epsilon)$ is constant.

Moral of the story: As long as the terms in the Sommerfeld expansion do not all vanish, the Sommerfeld expansion should be doing a good job! In particular, the Sommerfeld expansion result for the specific heat, $c_V = (\pi^2/3)k_B^2 T g(\epsilon_F)$, should be good even for the 2D fermi gas, and so also for the 2D fermi gas in a harmonic potential.

3) Two dimensional fermi gas in a hormonic potentral E(nx, ny) = two (nx+ny+1) gives the single particle energy levels We can represent the allowed surjee particle states by paints in the (n_x, n_y) plane with $n_x, n_y \ge 0$ Note the spacing between allowed points is An=1, always stays fixed! This is unlike particle in a box states, represented by points in a space, where the spacing between allowed points is $\Delta k = 2TT \longrightarrow 0$ as $L \gg \infty$ • • • n_X 0 in the thermody namic limit. The contours of constant every in (nx, ny) space are quien by $E = \hbar \omega_0 (n+1)$ where $n = n_X + n_Y$ for nx+ny =n constail nx Shaded triangle, divided by the area per illowed energy level (Anx)(Any) = 1. We should than multiply by the spin degeneracy factor $g_s = 2$ for the two possible spin states the The total number of single particle states with everygy less than E is therefore $G(E) = (Z)(\frac{1}{2}n^2) = n^2 = \left(\frac{E}{\pi\omega_0} - 1\right)^2$

Note - for G(E) above to be good, it is necessary that E is large so that n=nx+ny >>1 is large, so that the discreteness of the points in (nx,ny) space does not matter. Clearly if E was too small we would have a scheation like in this Sketch below, and then the area under the contour is not a good estimate of the muber of points inder 1919 the contour, nx 1 nx We will ignive this complication and use the above G(E) for all E. The convection to the quatities we calculate below will be small provided the number of particles is N≫), a) We can now compute the density of states per unit energy as $q(\varepsilon) = \frac{dG}{d\varepsilon} = \frac{2}{\pi w_0} \left(\frac{\varepsilon}{\pi w_0} - 1 \right)$ Note - the above is the density of states per energy and not the density of states per margy per volume, as was asked for in the problem. The latter would be just g(E)/V. But since the gas is confined by the has mouric potential and not by a box of given volume V, then there really is no V, and so it is more reasonable to use the above g/E) as the density of states per evergy. In terms of the above g(s), the expressions we will use in the later parts are: For the number of particles and total energy at T=0, N= SZFJE J(E) Here Emin = two is the lowest $E = \int_{\varepsilon_{min}}^{\varepsilon_{p}} d\varepsilon \, g(\varepsilon) \, \varepsilon$ allowed smile particle energy.

b) The Family energy is determined by

$$N = \int_{c_{min}}^{C_{F}} de g(E) = G(E_{F}) = \left(\frac{E_{F}}{\pi N_{0}} - 1\right)^{2}$$
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digto particle energy eigenstates at
$$E_F$$

 $E_F = \left(\frac{p^2}{2m} + \frac{1}{2}MW_0^2R^2\right)$
 $\frac{1}{K}$
 $\frac{1}{K}$

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same as for a So C~ <u>JE</u>~ 2 g(EF) kBT normal metal. The only difference is in Note, since g(E) here is the density of states per energy, and not the density the expression for g[EF] of states per energy per volume The above C is the total specific heat, and not the specific heat per volume $g(\varepsilon) = \frac{2}{\pi \omega_0} \left(\frac{\varepsilon}{\pi \omega_0} - 1 \right)$ From (a) $\mathcal{E}_{\mathbf{F}} = \pi \omega_0 \left(\mathbf{1} + \mathbf{1} \overline{\mathbf{N}} \right)$ From (b) $q(\varepsilon_{\rm F}) = \frac{2}{\pi \omega_{\rm D}} \sqrt{N}$ 50 $C \sim 4 \sqrt{N k_{B}^{2} T}$ $\frac{1}{\pi W_{0}}$ Note, in order to use the approximation that the number of fermions that get thermally excited is g(EF) kBT we should have by T >> tiwo, so that we do not have to worry about the discreteness of the spacing between the single particle energy levels,