PHYS 418 Solutions Problem Set 10

1) $\varepsilon = A|\vec{p}|^s = A\hbar^s|\vec{k}|^s = A\hbar^s k^s$

First lets compute the density of states per volume

We start by computing the total number of states per volume with energy less than ε. As we've done before

$$G(\varepsilon) = \frac{V_d k^d}{(2\pi/L)^d V}$$

where $V = L^d$ is the system volume

V_d = volume of sphere of unit radius in d-dimensions

Then

$$g(\varepsilon) = \frac{dG}{d\varepsilon} = \frac{dG}{dk}\frac{dk}{d\varepsilon} \qquad k = \left(\frac{\varepsilon}{A\hbar^s}\right)^{1/s}$$

$$= \frac{d V_d k^{d-1}}{(2\pi)^d} \frac{1}{s}\left(\frac{\varepsilon}{A\hbar^s}\right)^{\frac{1}{s}-1} \frac{1}{A\hbar^s}$$

$$= \frac{d V_d}{(2\pi)^d s} \frac{1}{A\hbar^s}\left(\frac{\varepsilon}{A\hbar^s}\right)^{\frac{d-1}{s}}\left(\frac{\varepsilon}{A\hbar^s}\right)^{\frac{1}{s}-1}$$

$$= \frac{d V_d}{(2\pi)^d s} \frac{1}{(A\hbar^s)^{d/s}} \varepsilon^{\frac{d}{s}-1}$$

$$\boxed{g(\varepsilon) = C\,\varepsilon^{\frac{d}{s}-1}}$$

where $C = \frac{d V_d}{(2\pi)^d s}\frac{1}{(A\hbar^s)^{d/s}}$ is a constant

a) The density of particles n is given by

$$n = \frac{n(0)}{V} + \int_0^\infty d\varepsilon\, g(\varepsilon)\langle n(\varepsilon)\rangle$$

$$= \underbrace{\frac{n(0)}{V}}_{\text{contrib from ground state}} + \underbrace{C\int_0^\infty d\varepsilon \frac{\varepsilon^{\frac{d}{s}-1}}{z^{-1}e^{\beta\varepsilon} - 1}}_{\text{contrib from excited states}}$$

let $y = \beta \varepsilon$ so $\varepsilon = k_B T y$

Then

$$n = \frac{n(0)}{V} + c(k_B T)^{d/s} \int_0^\infty dy \frac{y^{\frac{d}{s}-1}}{z^{-1}e^{y} - 1}$$

$$= \frac{n(0)}{V} + c(k_B T)^{d/s} \Gamma\left(\tfrac{d}{s}\right) g_{d/s}(z)$$

where $g_{d/s}(z)$ is the boson standard function of Notes 3-8. We have $g_{d/s}(z) = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^{d/s}}$ is a monotonic increasing function of z.

Because $n(0) = \frac{1}{z^{-1} - 1} = \frac{z}{1-z}$ must be positive

$\Rightarrow z \leq 1$ so the integral is largest when $z = 1$

If $g_{d/s}(z)$ diverges as $z \to 1$, then we can always choose an appropriate $z < 1$ so that the density of particles in excited states, given by the integral, will equal the total density n. Then $\frac{n(0)}{V} = \frac{z}{1-z}\frac{1}{V} \to 0$ as $V \to \infty$

and there will be no Bose-Einstein condensation

However, if $g_{d/s}(1)$ is finite, then the contribution of the excited states, given by the integral $\sim T^{d/s}$, will ultimately fall below n as T decreases, and we will have to have a $z = 1$ and a finite $\frac{n(0)}{V}$. There will then be B-E condensation!

So we want to evaluate the integral at $z = 1$ and see if it converges ($\Rightarrow$ will be B-E condensation) or diverges ($\Rightarrow$ will not be B-E condensation).

Consider this integral near its upper limit $y \to \infty$

Then $\int^{\infty} dy \, \frac{y^{\frac{d}{s}-1}}{e^{y}-1} \sim \int^{\infty} dy \, y^{\frac{d}{s}-1} e^{-y}$

will converge as the upper limit $\to \infty$

Now, consider the integral near its lower limit $y \to 0$

Then $\int_0 dy \, \frac{y^{\frac{d}{s}-1}}{e^{y}-1} \sim \int_0 dy \, \frac{y^{\frac{d}{s}-1}}{y} = \int_0 dy \, y^{\frac{d}{s}-2}$

$\sim y^{\frac{d}{s}-1} \Big|_0$

This will only converge as the lower limit $\to 0$ if $\frac{d}{s} - 1 > 0$ or $\boxed{d > s}$

So there is only B-E condensation if $d > s$.

For non-relativistic particles, $\varepsilon = \frac{p^2}{2m}$ and $s = 2$

So there can be B-E condensation only for $d > 2$. So there will be no B-E condensation in $d = 2$ dimensions.

b) When there is B-E condensation, for $T \leq T_c$ we know $z = 1$, and the density of particles in the excited states (the "normal" density) is given by

$$n - n_0 = C (k_B T)^{d/s} \, \Gamma(d/s) \, g_{\frac{d}{s}}(1)$$

$$= \tilde{C} \, T^{d/s} \quad \text{where } \tilde{C} \text{ is a constant}$$

Exactly at T_c, $n - n_0 = n = \tilde{C} \, T_c^{d/s}$

so $\boxed{T_c = \left(\frac{n}{\tilde{C}}\right)^{s/d}}$ $\qquad \tilde{C} = \frac{n}{T_c^{d/s}}$

For $T < T_c$ we then have for the condensate density

$$n_0 = \frac{N(0)}{V} = n - \tilde{C} \, T^{d/s}$$

$$= n - \frac{n}{T_c^{d/s}} \, T^{d/s}$$

$$\boxed{n_0 = n\left(1 - \left(\frac{T}{T_c}\right)^{d/s}\right)}$$

c) The average energy density is given by

$$\frac{E}{V} = \int_0^\infty d\varepsilon\, g(\varepsilon) \langle n(\varepsilon)\rangle \varepsilon$$

$$= C \int_0^\infty d\varepsilon \frac{\varepsilon^{\frac{d}{s}-1} \varepsilon}{z^{-1} e^{\beta\varepsilon} - 1}$$

substitute $y = \beta\varepsilon \Rightarrow \varepsilon = k_B T y$

$$\boxed{\frac{E}{V} = C(k_B T)^{\frac{d}{s}+1} \int_0^\infty dy \frac{y^{\frac{d}{s}}}{z^{-1} e^{y} - 1}}$$

The pressure is given by

$$\frac{P}{k_B T} = \frac{1}{V} \ln \mathcal{L} = -\int_0^\infty d\varepsilon\, g(\varepsilon) \ln(1 - z e^{-\beta\varepsilon})$$

$$= -C \int_0^\infty d\varepsilon\, \varepsilon^{\frac{d}{s}-1} \ln(1 - z e^{-\beta\varepsilon})$$

$y = \beta\varepsilon$

$$= -C(k_B T)^{d/s} \int_0^\infty dy\, y^{\frac{d}{s}-1} \ln(1 - z e^{-y})$$

integrate by parts

$$\frac{P}{k_B T} = \left[-C(k_B T)^{d/s} \frac{s}{d} y^{d/s} \ln(1 - z e^{-y})\right]_0^\infty$$

$$+ C(k_B T)^{d/s} \frac{s}{d} \int_0^\infty dy \frac{y^{d/s}\, z e^{-y}}{1 - z e^{-y}}$$

The first term is the boundary term, and it vanishes at its two limits. We can rewrite the integral term as

$$\frac{P}{k_B T} = c\,(k_B T)^{d/s}\,\frac{s}{d}\int_0^\infty dy\,\frac{y^{d/s}}{z^{-1}e^{y}-1}$$

So $$\boxed{P = c\,(k_B T)^{\frac{d}{s}+1}\,\frac{s}{d}\int_0^\infty dy\,\frac{y^{d/s}}{z^{-1}e^{y}-1}}$$

Comparing to the expression above for $\frac{E}{V}$ we have

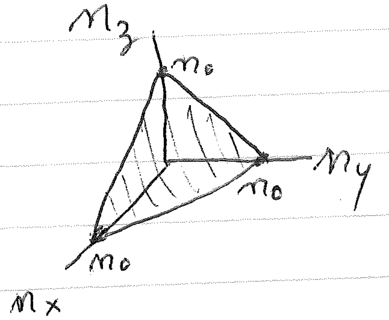
$$\boxed{P = \frac{s}{d}\,\frac{E}{V}}$$

d) No! There can be no B-E condensation for photons in any dimension, since photon number is <u>not</u> conserved! For photons $\mu = 0$ or $z = 1$ <u>always</u> and the temperature determines the average photon density, which is therefore <u>not fixed</u>.

2) a) $\mathcal{E} = \hbar\omega_0 (n_x + n_y + n_z + 3/2)$

Define $n_0 = \frac{\mathcal{E}}{\hbar\omega_0} - 3/2$

Then the states with energy less than or equal to $\mathcal{E}$ all lie below the surface;



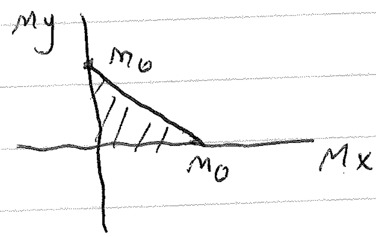
Since the spacing between allowed values of n_x, n_y, n_z is $\Delta n = 1$, the volume per allowed state in (n_x, n_y, n_z) space is $(\Delta n)^3 = 1$. So the number of states below the constant energy surface is just the volume under the surface. The surface is defined by the equation

$$n_x + n_y + n_z = n_0 \quad \text{or} \quad n_z = n_0 - n_x - n_y$$

So to find the volume we just integrate

$\int dn_x \int dn_y \; n_z(n_x, n_y)$

where the region of integration for n_x and n_y is



So, volume under surface = number of states with energy less than or equal to ε, this is the definition of $G(\varepsilon)$

$$G(\varepsilon) = \int_0^{m_0} dm_x \int_0^{m_0 - m_x} dm_y \, (m_0 - m_x - m_y)$$

$$= \int_0^{m_0} dm_x \left[m_0(m_0 - m_x) - m_x(m_0 - m_x) - \frac{1}{2}(m_0 - m_x)^2 \right]$$

$$= m_0^3 - \frac{1}{2} m_0^3 - \frac{1}{2} m_0^3 + \frac{1}{3} m_0^3 - \frac{1}{6} m_0^3$$

$$= m_0^3 \left[1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{6} \right] = \frac{1}{6} m_0^3$$

$$G(\varepsilon) = \frac{1}{6}\left(\frac{\varepsilon}{\hbar\omega_0} - \frac{3}{2}\right)^3$$

So density of states is

$$\boxed{g(\varepsilon) = \frac{dG}{d\varepsilon} = \frac{1}{2}\left(\frac{\varepsilon}{\hbar\omega_0} - \frac{3}{2}\right)^2 \frac{1}{\hbar\omega_0}}$$

b) The chemical potential must satisfy $\mu \leq \varepsilon_{min}$ where ε_{min} is the lowest energy level of the system. This is necessary so that the Bose occupation function

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$$ is always positive for any allowed energy ε

Here $\varepsilon_{min} = \frac{3}{2}\hbar\omega_0$ ie when $n_x = n_y = n_z = 0$

so $\mu \le \varepsilon_{min} \Rightarrow z = e^{\beta\mu} \le e^{\beta\varepsilon_{min}}$

largest value of z is $\boxed{z = e^{\frac{3}{2}\beta\hbar\omega_0}}$

c) The number of particles in the system can be written as

$$N = N_0 + \int_{\varepsilon_{min}}^{\infty} d\varepsilon \, \frac{g(\varepsilon)}{z^{-1}e^{\beta\varepsilon} - 1}$$

where No is the number in the ground state with energy ε_{min}. To see if there is Bose-Einstein condensation, we look at the integral term and see whether or not it diverges when z takes its maximal possible value. If it diverges, then it is always possible to choose a $z < z_{max}$ so that the integral gives the total number of particles N and so $N_0 = 0$. In this case there will be no B-E condensation. But if the integral converges to a finite value, then for sufficiently small T this value will fall below N and then the only way to satisfy the above equation is to have a finite N_0, ie there will be B-E condensation, with a finite fraction N_0/N of the particles in the ground state.

When $z = z_{max} = e^{\beta \varepsilon_{min}}$ the integral is

$$I = \int_{\varepsilon_{min}}^{\infty} d\varepsilon \, \frac{g(\varepsilon)}{e^{\beta(\varepsilon - \varepsilon_{min})} - 1}$$

change integration variable to $x = \beta(\varepsilon - \varepsilon_{min})$

$$I = \frac{1}{\beta} \int_0^{\infty} dx \, \frac{g\left(\frac{x}{\beta} + \varepsilon_{min}\right)}{e^x - 1} = \frac{1}{\beta \hbar \omega_0} \frac{1}{2} \int_0^{\infty} dx \left(\frac{x}{\beta \hbar \omega_0}\right)^2 \frac{1}{e^x - 1}$$

$$= \left(\frac{k_B T}{\hbar \omega_0}\right)^3 \frac{1}{2} \int_0^{\infty} dx \, \frac{x^2}{e^x - 1}$$

Using the zeta function $\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dx \, \frac{x^{s-1}}{e^x - 1}$

and $\Gamma(3) = 2! = 2$, we have

$$I = \left(\frac{k_B T}{\hbar \omega_0}\right)^3 \zeta(3)$$

the integral converges!
so there is B-E condensation!

d) At T_c, the number of excited particles is equal to the total number of particles and $z = z_{max}$. So from part (c) we get

$$N = \left(\frac{k_B T_c}{\hbar \omega_0}\right)^3 \zeta(3) \quad \text{or} \quad \boxed{\left(\frac{N}{\zeta(3)}\right)^{1/3} \frac{\hbar \omega_0}{k_B} = T_c}$$

e) For $T < T_c$ the number of particles in the ground state is

$$N_0 = N - I = N - \left(\frac{k_B T}{\hbar\omega_0}\right)^3 \zeta(3)$$

$$= N - N\left(\frac{T}{T_c}\right)^3 \qquad \text{from (d)}$$

$$\boxed{N_0(T) = N\left(1 - \left(\frac{T}{T_c}\right)^3\right)}$$

3) let $B \equiv A_2$ the boson.

reaction is $\quad 2A \leftrightarrow B$

condition of chemical equilib requires: $\boxed{2\mu_A = \mu_B}$

if we call $\mu_A \equiv \mu$, then $\mu_B = 2\mu$

(above follows from maximizing entropy $S(E, N_A, N_B)$

$$dS = \frac{\partial S}{\partial N_A} dN_A + \frac{\partial S}{\partial N_B} dN_B = -\frac{1}{T}\left(\mu_A dN_A + \mu_B dN_B\right) = 0$$

Conservation of total number of A requires $N_A + 2N_B = \text{const}$

So $dN_A + 2dN_B = 0 \Rightarrow dN_A = -2dN_B$.

$$\Rightarrow dS = -\frac{1}{T}\left(-2\mu_A + \mu_B\right)dN_B = 0 \rightarrow 2\mu_A = \mu_B \quad)$$

<u>For fermions A</u>: $\epsilon_i = \frac{p_i^2}{2m_A}$ spin degeneracy $g_s = 2$

$$\frac{N_A}{V} = \frac{1}{V}\sum_i \frac{1}{z_A^{-1} e^{\beta\epsilon_i} + 1} = \frac{2\, g_s}{\sqrt{\pi}\,\lambda_A^3}\int_0^\infty dy\, \frac{y^{1/2}}{z_A^{-1} e^{y} + 1}$$

(1) $\boxed{\frac{N_A}{V} = \frac{2}{\lambda_A^3} f_{3/2}(z_A)}$ where $\lambda_A = \left(\frac{h^2}{2\pi m_A k_B T}\right)^{1/2}$

<u>For bosons B</u>: $\epsilon_i = \frac{p_i^2}{2m_B} + \epsilon_0$ spin degeneracy $g_s = 1$

$\leftarrow$ cost ϵ_0 to create boson

$$\frac{N_B}{V} = \frac{1}{V}\sum_i \frac{1}{z_B^{-1} e^{\beta\epsilon_i} + 1} = \frac{1}{V}\sum_i \frac{1}{z_B^{-1} e^{\beta\epsilon_0}\, e^{\beta p_i^2/2m} + 1}$$

$$= \frac{1}{V}\sum_i \frac{1}{\tilde{z}_B^{-1}\, e^{\beta p_i^2/2m} + 1} = n_0 + \frac{2 g_s}{\sqrt{\pi}\,\lambda_B^3}\int_0^\infty dy\, \frac{y^{1/2}}{\tilde{z}_B^{-1} e^{y} - 1}$$

(2) $$\boxed{\frac{N_B}{V} = n_0 + \frac{1}{\lambda_B^3} g_{3/2}(\bar{z}_B)}$$

where n_0 is the condensate density and $\bar{z}_B^{-1} \equiv z_B^{-1} e^{\beta\epsilon_0}$

or $\bar{z}_B = z_B e^{-\beta\epsilon_0} = e^{\beta(\mu_B-\epsilon_0)} = e^{\beta(2\mu_A-\epsilon_0)}$

Eqns (1) and (2) have two unknowns: μ and n_0

where $z_A = e^{\beta\mu}$ and $\bar{z}_B = e^{\beta(2\mu-\epsilon_0)}$

needed to determine the densities $\frac{N_A}{V}$ and $\frac{N_B}{V}$.

We have one constraint: (3) $\boxed{N_A + 2N_B = N}$ initial given number of A

We need a second constraint – we get this by condition for Bose Einstein condensation:

For (4) $\boxed{\begin{array}{ll} T > T_c, & n_0 = 0 \\ T < T_c, & \bar{z}_B = 1 \end{array}}$ no condensate density $\Rightarrow \mu = \epsilon_0/2$

The combination of equations (1) and (2) with constraints (3) and (4) is in principle sufficient to determine N_A and N_B at any temperature T, as follows

(3) + (1) + (2) $\Rightarrow$ $$\frac{N}{V} = \frac{2}{\lambda_A^3} f_{3/2}(z_A) + 2n_0 + \frac{2}{\lambda_B^3} g_{3/2}(\bar{z}_B)$$

use $z_B = e^{\beta 2\mu_A} = z_A^2$

$\bar{z}_B = z_B e^{-\beta\epsilon_0} = z_A^2 e^{-\beta\epsilon_0}$

$z_A = \bar{z}_B^{1/2} e^{\beta\epsilon_0/2}$

$$\Rightarrow \boxed{\frac{N}{V} = \frac{2}{\lambda_A^3} f_{3/2}\left(\bar{z}_B^{1/2} e^{\beta\epsilon_0/2}\right) + 2n_0 + \frac{2}{\lambda_B^3} g_{3/2}(\bar{z}_B)}$$

two unknowns, $\bar{z}_B$ and n_0.

try to find a solution of above with $n_0 = 0$ and $\mathcal{Z}_B < 1$. If possible then this is the solution. Knowing $\mathcal{Z}_B$ then lets one compute N_A and N_B. If no such solution exists, then set $\mathcal{Z}_B = 1$ and solve for n_0. Knowing $\mathcal{Z}_B = 1$ and n_0 then allows one to solve for N_A and N_B.

At $T=0$

We know that all the bosons B will be condensed in the ground state, hence $\mathcal{Z}_B = 1$ and $n_0 = \frac{N_B}{V}$.

$$\Rightarrow \quad \mathcal{Z}_A = \mathcal{Z}_B^{-1/2} e^{\beta\epsilon_0/2} = e^{\beta\epsilon_0/2} \quad \Rightarrow \mu_A = \epsilon_0/2 = \epsilon_F$$

fermi energy

$$\Rightarrow \quad \frac{N_A}{V} = \frac{g_s}{6\pi^2}\left(\frac{2m_A\epsilon_F}{\hbar^2}\right)^{3/2}$$

$\leftarrow$ relation between density and ϵ_F for $T=0$ fermi gas

$$= \frac{1}{3\pi^2}\left(\frac{m_A\epsilon_0}{\hbar^2}\right)^{3/2}$$

$$N_A + 2N_B = N \quad \Rightarrow \quad N_B = \frac{N-N_A}{2}$$

$$\text{So} \quad \frac{\#\text{ bosons}}{\#\text{ fermions}} = \frac{N_B}{N_A} = \frac{N}{2N_A} - \frac{1}{2} = \boxed{\frac{1}{2}\left(\frac{N}{V}\, 3\pi^2\left(\frac{\hbar^2}{m_A\epsilon_0}\right)^{3/2} - 1\right)}$$

* Note: if N_A as computed above turns out bigger than N, then $N_B = 0$ and $N_A = N$ – all the A's remain as fermions, filling up to a fermi energy ϵ_F given by

$$\frac{N}{V} = \frac{1}{3\pi^2}\left(\frac{2m_A\epsilon_F}{\hbar^2}\right)^{3/2}$$

The interpretation of the above result is clear:

One fills up fermion energy levels until one reaches the value $\epsilon_0/2$. Then, instead of adding the next two fermions to the fermi gas, which would cost energy $\frac{\epsilon_0}{2}^+ + \frac{\epsilon_0}{2}^+ > \epsilon_0$, it is energetically more favorable to create a boson A_2 in its ground state — this costs only ϵ_0. Similarly all further ~~states~~ fermions will go to create additional bosons in the condensate. If there are not enough fermions to reach a fermion energy level of $\frac{\epsilon_0}{2}$, then no bosons will be created.

Classically, at $T=0$, all the particles will go into the lowest energy state - this will be the $\vec{k}=0$ fermion level with $\epsilon=0$ (The boson energy levels have a minimum energy ϵ_0). Hence will will have only fermions

$$N_A = N, \quad N_B = 0, \quad \frac{N_B}{N_A} = 0$$