Solutions Problem Set 2

1) Prove $\left(\frac{\partial C_p}{\partial p}\right)_T = -TV\left[\alpha^2 + \left(\frac{\partial \alpha}{\partial T}\right)_p\right]$

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_p \quad = \text{definition of } C_p$$

$$S(T,p) = -\left(\frac{\partial G(T,p)}{\partial T}\right)_p$$

$$C_p = -T\left(\frac{\partial^2 G}{\partial T^2}\right)_p$$

$$\left(\frac{\partial C_p}{\partial p}\right)_T = -T\left(\frac{\partial^3 G}{\partial T^2 \partial p}\right) = -T\left(\frac{\partial^2}{\partial T^2}\left(\frac{\partial G}{\partial p}\right)_T\right)_p$$

$$= -T\left(\frac{\partial^2 V}{\partial T^2}\right)_p = -T\left(\frac{\partial}{\partial T}\left(\frac{\partial V}{\partial T}\right)_p\right)_p$$

$$= -T\,\frac{\partial}{\partial T}(V\alpha)_p \quad \text{- using definition of } \alpha$$

$$= -TV\left(\frac{\partial \alpha}{\partial T}\right)_p - T\alpha\left(\frac{\partial V}{\partial T}\right)_p$$

$$= -TV\left(\frac{\partial \alpha}{\partial T}\right)_p - T\alpha V\alpha$$

$$\boxed{\left(\frac{\partial C_p}{\partial p}\right)_T = -TV\left[\alpha^2 + \left(\frac{\partial \alpha}{\partial T}\right)_p\right]}$$

2) $T = \left(\frac{V}{V_0}\right)^{\eta} T_0$ gives temperature as system changes volume V

a) Work done on the gas

$$dW = -p\,dV$$

use ideal gas $pV = N k_B T \Rightarrow p = \frac{N k_B T}{V}$

$$dW = -N k_B T \frac{dV}{V}$$

total work done is

$$W = \int dW = -N k_B \int_{V_0}^{V_1} \frac{dV}{V} T(V)$$

$$= N k_B \int_{V_0}^{V_1} \frac{dV}{V} \left(\frac{V}{V_0}\right)^{\eta} T_0 = -N k_B T_0 \int_{V_0}^{V_1} dV \frac{V^{\eta-1}}{V_0^{\eta}}$$

$$= \frac{-N k_B T_0}{V_0^{\eta}} \frac{V_1^{\eta} - V_0^{\eta}}{\eta} = \frac{N k_B T_0}{\eta}\left(1 - \left(\frac{V_1}{V_0}\right)^{\eta}\right)$$

$$\boxed{W = \frac{N k_B T_0}{\eta}\left[1 - \left(\frac{V_1}{V_0}\right)^{\eta}\right]}$$

b) use $E = \frac{3}{2} N k_B T$ for ideal gas

$$\boxed{\Delta E = \frac{3}{2} N k_B (T_1 - T_0) = \frac{3}{2} N k_B T_0 \left[\left(\frac{V_1}{V_0}\right)^{\eta} - 1\right]}$$

c) Heat transfer to the gas

$$dQ \equiv TdS = dE - dW \qquad \left(\text{from } dE = TdS - p\,dV \right.$$
$$\Rightarrow \Delta Q = \Delta E - \Delta W \qquad \left.\text{and } dW = -p\,dV\right)$$

$$\Delta Q = N k_B T_0 \left[\left(\frac{V_1}{V_0}\right)^{\eta} - 1\right]\frac{3}{2} - \frac{N k_B T_0}{\eta}\left[1 - \left(\frac{V_1}{V_0}\right)^{\eta}\right]$$

$$\boxed{\Delta Q = N k_B T_0 \left[\left(\frac{V_1}{V_0}\right)^{\eta} - 1\right]\left[\frac{3}{2} + \frac{1}{\eta}\right]}$$

$$\Delta Q = 0 \quad \text{when} \quad \frac{3}{2} + \frac{1}{\eta} = 0 \quad \Rightarrow \boxed{\eta = -\frac{2}{3}}$$

From problem ① if the gas expands adiabatically (ie $\Delta S = 0 \Rightarrow \Delta Q = 0$) then

$$p V^{5/3} = \text{const}$$

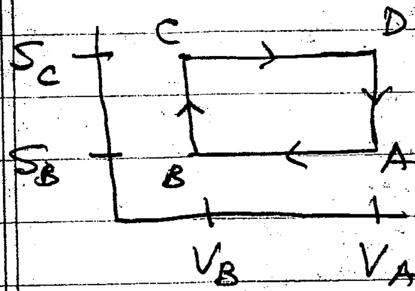
using $pV = N k_B T$ the above becomes

$$\frac{N k_B T}{V} V^{5/3} = N k_B T V^{2/3} = \text{const}$$

$$\text{or} \quad T \sim V^{-2/3}$$

which gives $\eta = -2/3$ in agreement with part (c)

3) Otto cycle



heat extracted in $B \to C$
work done by system in $C \to D$
heat returned in $D \to A$
work done on system in $A \to B$

efficiency $\epsilon \equiv \dfrac{Q_{BC} - Q_{AD}}{Q_{BC}} = 1 - \dfrac{Q_{AD}}{Q_{BC}}$

using $dQ = TdS$, the heat extracted from reservoir and added to the system in step $B \to C$ is

$$Q_{BC} = \int_B^C T\,dS$$ integral is taken at constant V

Now we know $T = \left(\frac{\partial E}{\partial S}\right)_{V,N}$ so

$$Q_{BC} = \int_B^C \left(\frac{\partial E}{\partial S}\right)_{V,N} dS = E_C - E_B$$

(we can do the integral since trajectory of integration is at constant V. or more generally, $TdS = dE + p\,dV$, but here $dV = 0$)

For ideal gas, as assumed here, $E = \frac{3}{2} N k_B T$

So $Q_{BC} = \frac{3}{2} N k_B (T_C - T_B)$

similarly

$$Q_{AD} = \frac{3}{2} N k_B (T_D - T_A)$$

efficiency

$$\epsilon = 1 - \left(\frac{T_D - T_A}{T_C - T_B}\right)$$

Finally we need to relate T_A, T_B, T_C, T_D to the volumes V_A and V_B

From Set 1, problem 4 we had for the adiabatic expansion of the ideal gas,

$$pV^{5/3} = \text{const}$$

using $pV = Nk_BT$ gives $Nk_BTV^{2/3} = \text{const}$

So along path $C \to D$ we get

$$T_C V_B^{2/3} = T_D V_A^{2/3} \Rightarrow T_D = T_C \left(\frac{V_B}{V_A}\right)^{2/3}$$

and along path $B \to A$ we get

$$T_B V_B^{2/3} = T_A V_A^{2/3} \Rightarrow T_A = T_B \left(\frac{V_B}{V_A}\right)^{2/3}$$

So

$$\epsilon = 1 - \left[\frac{T_C \left(\frac{V_B}{V_A}\right)^{2/3} - T_B \left(\frac{V_B}{V_A}\right)^{2/3}}{T_C - T_B}\right]$$

$$\epsilon = 1 - \left(\frac{V_B}{V_A}\right)^{2/3}$$

Now for ideal gas

$$C_V = \frac{3}{2} N k_B$$

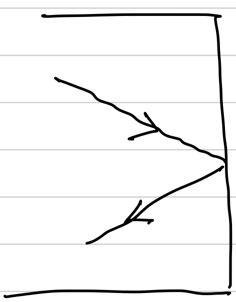
$$C_p = \frac{5}{2} N k_B$$

$$\frac{C_p - C_V}{C_V} = \frac{5-3}{3} = \frac{2}{3}$$

$$\boxed{e = 1 - \left(\frac{V_B}{V_A}\right)^{(C_p - C_V)/C_V}}$$

4) N particles in a box of volume V travel with extremely relativistic speeds such that we have $|\vec{v}| \approx c$ and their energy can be taken to be $\varepsilon = pc$, with p the magnitude of the relativistic momentum.

Consider such a particle colliding with the wall. We assume the collision is elastic.



Then, just like in notes 2-1 for non-relativistically moving particles, we can write the pressure on the wall as

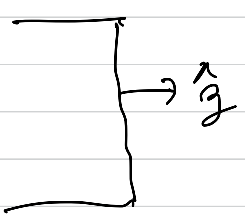
$$\mathbb{P} = \left\langle \frac{(\Delta p_\perp)\,\text{rate}}{A} \right\rangle$$

($\mathbb{P}$ is pressure
p is momentum.

where $\Delta p_\perp = 2p_\perp$ is the change in momentum of the particle during the collision, $p_\perp$ is the component of momentum in the direction perpendicular to the wall, A is the wall area, and the rate of collisions is

$$\text{rate} = \frac{1}{2}\frac{N}{V} A\, v_\perp$$

with $v_\perp$ the component of the velocity in the direction perpendicular to the wall.



If the normal vector to the wall is $\hat{z}$, then $p_\perp = p_z$ and $v_\perp = v_z$

$\langle \cdots \rangle$ represents the average over all particles and all collisions

$$\Rightarrow \quad \mathbb{P} = \frac{N}{V} \langle p_\perp v_\perp \rangle$$

Now, by our assumption that particles travel equally likely in all directions we, can write

$$\langle p_\perp v_\perp \rangle = \langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_z \rangle$$
$$= \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle = \frac{1}{3} \langle p v \rangle$$

The last step, $\vec{p} \cdot \vec{v} = pv$, is because $\vec{p}$ and $\vec{v}$ are colinear. Now we $v \simeq c$ and we get

$$\mathbb{P} = \frac{1}{3} \frac{N}{V} \langle p c \rangle = \frac{1}{3} \frac{N}{V} \langle \varepsilon \rangle$$

$$\Rightarrow \langle \varepsilon \rangle = \frac{3 \mathbb{P} V}{N}$$

Now use the ideal gas law $\mathbb{P} V = N k_B T$ to get

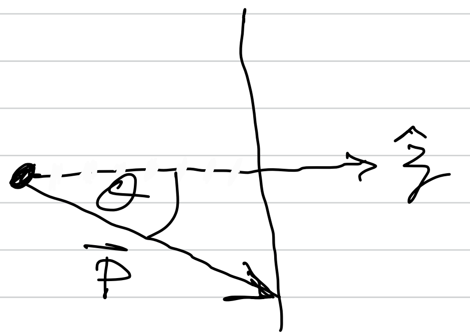
$$\boxed{\langle \varepsilon \rangle = 3 k_B T}$$

compare that to the non-relativistic result $\langle \varepsilon \rangle = \frac{3}{2} k_B T$

To further convince you that

$$\langle p_\perp v_\perp \rangle = \frac{1}{3}\langle p v \rangle$$

we can write



$$p_\perp = p\cos\theta$$

$$v_\perp = v\cos\theta$$

$$\langle p_\perp v_\perp \rangle = \langle p v \cos^2\theta \rangle$$

The average $\langle \cdots \rangle$ is an average over all particle speeds v, and all velocity directions, specified by the spherical angles (φ, θ).

We can first do the average over all directions to get

$$\langle \cos^2\theta \rangle = \underbrace{\frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta}_{\text{average over all spherical angles}} \cos^2\theta$$

$$\langle \cos^2\theta \rangle = \frac{1}{2}\left[\frac{-\cos^3\theta}{3}\right]_0^{\pi} = \frac{1}{2}\left[\frac{1}{3}+\frac{1}{3}\right] = \frac{1}{3}$$

so $\langle p_\perp v_\perp \rangle = \langle p v \cos^2\theta \rangle = \frac{1}{3}\langle p v \rangle$

Note: Nowhere in our calculation did we need to use the relativistic expression for momentum

$p = m\gamma v$ with $\gamma = \dfrac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$