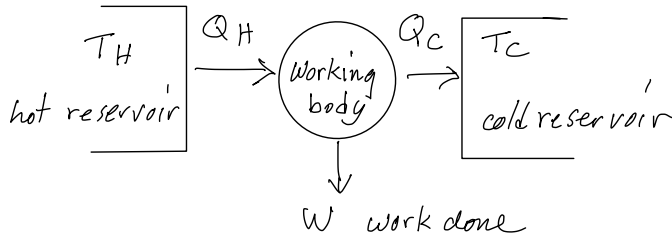


## Unit 1-10: Heat Engines

One of the most important topics that propelled early investigations of thermodynamics was the study of *heat engines* – how to extract mechanical work from thermal energy, i.e. heat. Heat engines are the subject of this section, and so here is your chance to finally understand the Carnot cycle!

A model for a heat engine is sketched below.



Two thermal reservoirs, one hot at temperature  $T_H$  and one cold at temperature  $T_C$ , interact with a “working body” that will convert heat flowing from the hot to the cold reservoir into mechanical work.

For example, the hot reservoir might be some sort of furnace burning fuel. The cold reservoir might be the ambient atmosphere. The working body might be a chamber of gas that can push a piston.

The hot reservoir can be called the “heat source”. The cold reservoir can be called the “heat sink.”

The working body is assumed to run on a “thermodynamic cycle” – it passes quasistatically (at every moment in time the system is in thermal equilibrium) through a set of thermodynamic parameters, periodically returning to the same initial state in order to once again repeat the process for another cycle. This assumption of quasistatic behavior is clearly a gross simplification for a real engine, but we make it as a theoretical model.

The *thermodynamic efficiency* of the engine is,

$$\boxed{\varepsilon = \frac{W}{Q_H} = \frac{\text{work output}}{\text{heat input}}} \quad \text{in one cycle.} \quad (1.10.1)$$

$\varepsilon$  measures the fraction of the heat  $Q_H$  pumped into the working body that gets converted into mechanical work  $W$  done by the working body. Clearly  $\varepsilon = 1$  would be ideal efficiency – all heat is converted into mechanical work. We will see that this ideal can never be attained.

Since the working body operates quasistatically, changes in energy are related to changes in the thermodynamic parameters. For a gas, this would be,

$$dE = TdS - pdV = dQ - dW = \text{heat in} - \text{work out} \quad (1.10.2)$$

(note, the dash on the differential,  $d$ , indicates that the change in the quantity may depend on the path taken.)

Since the working body operates in a cycle, returning to its initial state, we can integrate the above over one cycle of operation to get,

$$0 = \oint_{\text{cycle}} dE = \oint_{\text{cycle}} TdS - \oint_{\text{cycle}} pdV \quad (1.10.3)$$

where  $\oint_{\text{cycle}} dE = E_{\text{final}} - E_{\text{initial}} = 0$  since the system returns to its initial state after one complete cycle.

Now  $\oint_{\text{cycle}} p dV = W$  is the mechanical work done in one cycle.

And  $\oint_{\text{cycle}} T dS = Q_H - Q_C = Q^{\text{total}}$  is the total heat transferred to the working body in one cycle.

Here  $Q_H$  is the heat pumped in from the hot reservoir, and  $Q_C$  is the heat released to the cold reservoir.

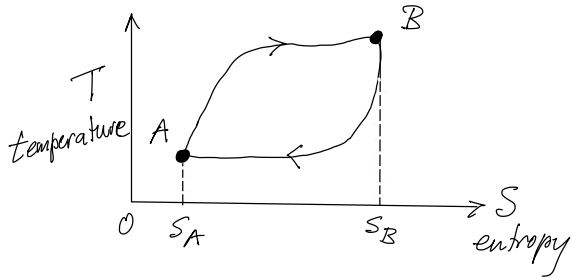
So we can write Eq. (1.10.3) as,

$$0 = Q_H - Q_C - W \quad \Rightarrow \quad W = Q_H - Q_C \quad (1.10.4)$$

The efficiency can then be written as,

$$\varepsilon = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H} \quad \Rightarrow \quad \boxed{\varepsilon = 1 - \frac{Q_C}{Q_H}} = 1 - \frac{\text{heat released}}{\text{heat absorbed}} \quad (1.10.5)$$

A general engine cycle can be represented as a closed loop in the  $T - S$  plane, representing the temperature and entropy of the working body. Because of the physical constraints of the engine (for example the temperatures of the reservoirs, or the minimum and maximum volumes of a piston) there is generally a minimum allowed value of  $S$  and  $T$  and a maximum allowed value of  $S$  and  $T$  that the engine can operate between. We can therefore take the cycle as going from the point  $A$  of minimum  $S$  to the point  $B$  of maximum  $S$ , and then returning to  $A$ , as in the sketch below.



As the system goes from  $A \rightarrow B$ , the heat transferred *to* the working body is the area under the top curve,

$$\Delta Q_{AB} = \int_A^B T dS > 0 \quad (1.10.6)$$

$\Delta Q_{AB} > 0$  means this is the heat transferred to the working body from the hot reservoir, i.e.  $Q_H = \Delta Q_{AB}$ .

As the system goes from  $B \rightarrow A$ , the heat transferred *from* the working body is the area under the bottom curve,

$$\Delta Q_{BA} = \int_B^A T dS < 0 \quad (1.10.7)$$

$\Delta Q_{BA}$  is negative since the integral goes from  $S_B$  to the smaller  $S_A$ .

$\Delta Q_{BA} < 0$  means this is the heat *released* by the working body to the cold reservoir,  $Q_C = -\Delta Q_{BA}$ .

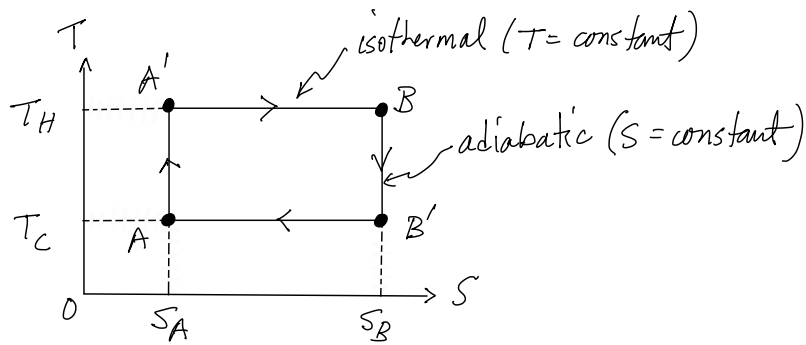
The total heat transferred,  $Q^{\text{tot}} = Q_H - Q_C$ , is thus the area bounded by the closed loop.

To maximize the engine efficiency, we want to maximize the area under the curve from  $A$  to  $B$  while minimizing the area under the curve from  $B$  to  $A$ .

$$\varepsilon = 1 - \frac{Q_C}{Q_H} = 1 - \frac{|\Delta Q_{BA}|}{\Delta Q_{AB}} = 1 - \frac{\text{area under } B \rightarrow A}{\text{area under } A \rightarrow B} \quad (1.10.8)$$

Now the loop must be such that all temperatures on it are in the range  $T_C \leq T \leq T_H$ , since the temperatures are bounded by the hot reservoir from above and by the cold reservoir from below.

For an engine operating between the two reservoirs of fixed temperatures  $T_H$  and  $T_C$ , the most efficient engine cycle is the *Carnot cycle* given by the  $T - S$  diagram below.



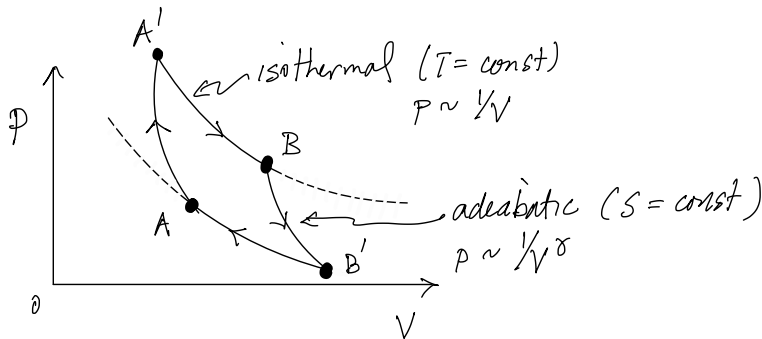
One can easily see that this rectangular loop maximizes the area under the  $A \rightarrow B$  part of the loop, while minimizing the area under the return  $B \rightarrow A$  part of the loop. For this cycle it is easy to compute the efficiency  $\varepsilon$ .

$$Q_H = T_H(S_B - S_A) = \text{heat absorbed}, \quad Q_C = T_C(S_B - S_A) = \text{heat released} \quad (1.10.9)$$

$$\boxed{\varepsilon = 1 - \frac{Q_C}{Q_H} = 1 - \frac{T_C}{T_H}} \quad (1.10.10)$$

Since  $T_C$  and  $T_H$  must be finite, one always has  $\varepsilon < 1$ .

For demonstrating that the Carnot cycle is the most efficient, it was convenient to show the cycle in the  $T-S$  plane. However one often sees it depicted instead as a loop in the  $p-V$  plane. There it looks as sketched below.



- 1)  $A' \rightarrow B$  is isothermal expansion of the gas at the temperature  $T_H$  of the hot reservoir. Expanding the gas pushes the piston and does work. The working body absorbs heat  $Q_H = T_H(S_B - S_A)$  from the hot reservoir.
- 2)  $B \rightarrow B'$  is the adiabatic expansion of the gas. The working body is thermally isolated from the hot reservoir. The gas continues to do work as it expands. No heat is added to the gas as it expands (i.e.  $\Delta Q = 0$ ) so the temperature of the gas decreases until it reaches  $T_C$ , the temperature of the cold reservoir.
- 3)  $B' \rightarrow A$  is isothermal compression of the gas. The working body is now in thermal contact with the cold reservoir, maintaining its temperature at  $T_C$ . Work is done *on* the working body to compress the gas keeping  $T_C$  constant, so the working body releases heat  $-\Delta Q = Q_C = T_C(S_B - S_A)$  to the cold reservoir.
- 4)  $A \rightarrow A'$  is adiabatic compression of the gas. The working body is thermally isolated from the reservoirs. No heat flows into or out of the working body. Compressing the gas raises its temperature back up to  $T_H$  of the hot reservoir, and the cycle begins again.

The *isothermal* segments of the cycle in the  $p-V$  plane are curves given by,

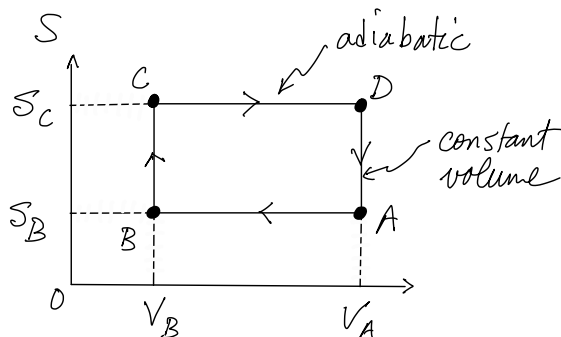
$$pV = Nk_B T \quad \Rightarrow \quad p = \frac{Nk_B T}{V} \quad \text{with } N \text{ fixed and } T \text{ fixed, one has } p \sim \frac{1}{V} \quad (1.10.11)$$

The *adiabatic* segments of the cycle in the  $p - V$  plane are curves given by,

$$p \sim \frac{1}{V^\gamma}, \quad \gamma > 0 \quad (1.10.12)$$

(see problem 4 on Problem Set 1)

Another common thermodynamic cycle is the *Otto cycle* which consists of two adiabatic and two constant volume “strokes,” as in the sketch below.



One can show that the efficiency of the Otto cycle is given by,

$$\varepsilon = 1 - \left(\frac{V_B}{V_A}\right)^{\frac{C_p - C_V}{C_V}} \quad (1.10.13)$$

where  $C_V$  and  $C_p$  are the specific heats at constant volume and constant pressure respectively.

The Otto cycle is a model for an automobile engine, where the piston cycles between two fixed volumes on each stroke of the engine.

Note: We can see that the same cycle can be represented as a curve in either the  $T - S$  plane,  $p - V$  plane,  $S - V$  plane, or other variable choices, by the following observation.

Let us take  $E(S, V)$  as the thermodynamic potential. A given “stroke” of a cycle can therefore be represented as a curve in the  $S - V$  plane (as in the Otto cycle diagram above), that tells how  $S$  must vary as  $V$  varies during the stroke of the cycle, i.e.  $S_0(V)$ .

But we also have

$$-\left(\frac{\partial E}{\partial V}\right)_S = p(S, V) \quad (1.10.14)$$

So the stroke can also be represented by the curve  $p(S_0(V), V)$  in the  $p - V$  plane (as in the  $p - V$  diagram for the Carnot cycle).

Also, we can invert the  $S_0(V)$  curve to get  $V_0(S)$ , and then use,

$$\left(\frac{\partial E}{\partial S}\right)_V = T(S, V) \quad (1.10.15)$$

to represent the stroke as the curve  $T(S, V_0(S))$  in the  $T - S$  plane (as in the  $T - S$  diagram for the Carnot cycle).

Hence we can represent the stroke in many different, but equivalent, ways.

## The Laws of Thermodynamics

Having finished our discussion of classical thermodynamics, it is perhaps worthwhile to cite the laws of thermodynamics in their more traditional form. You should convince yourself that these all follow from the postulates we have stated in our presentation.

### Zeroth Law

*If two systems are both in thermal equilibrium with a third system, then they are in thermal equilibrium with each other.*

### First Law

*In a closed system (where there is no transfer of matter into or out of the system), the change in internal energy of the system  $\Delta U$ , is given by the heat supplied to the system  $\Delta Q$  minus the work done by the system  $\Delta W$ . So  $\Delta U = \Delta Q - \Delta W$ .*

### Second Law

*When two initially isolated systems in separate but nearby regions of space, each in thermodynamic equilibrium with itself but not necessarily with each other, are then allowed to interact, they will eventually reach a mutual thermodynamic equilibrium. The sum of the entropies of the initially isolated systems is less than or equal to the total entropy of the final combination. Equality occurs just when the two original systems have all their respective intensive variables (temperature, pressure) equal; then the final system also has the same values.*

### Third Law

*A system's entropy approaches a constant value as its temperature approaches absolute zero. If the ground state of the system is unique, that constant is zero.*