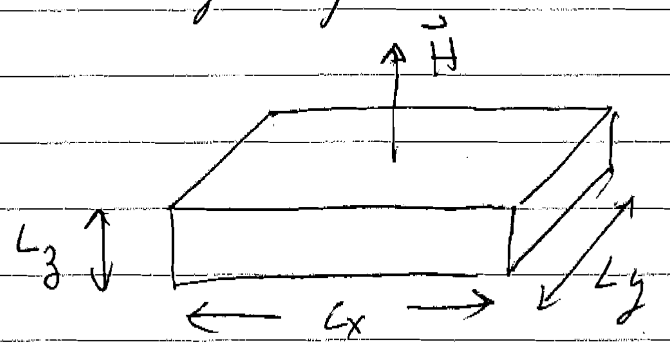
Landau Diamagnetism - Landau Levels

Here we wish to consider the effect of the magnetic field on the orbital motion of the conduction electrons. To do so we must solve the quantum mechanical problem of a charged particle moving in a uniform magnetic field.

The geometry we consider is



uniform $\vec{H} = H\hat{z}$

For a particle of charge q in a static uniform magnetic field, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2$$

where $\vec{A}$ is the vector potential, $\vec{H} = \nabla \times \vec{A}$

$q = -e$ is the charge of the electron

For $\vec{H} = H\hat{z}$ we will use $\vec{A} = -yH\hat{x}$

Substitute these into $\mathcal{H}$ to get:

$$\mathcal{H} = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c} y H \hat{x}\right)^2$$

$$= \frac{1}{2m}\left[-\hbar^2\frac{\partial^2}{\partial z^2} - \hbar^2\frac{\partial^2}{\partial y^2} + \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{e}{c} H y\right)^2\right]$$

We want to find the eigenstates ψ that solve

$$\mathcal{H}\psi = \varepsilon\psi \qquad \varepsilon \text{ is eigenvalue of energy}$$

Try solution of the form

$$\psi(x,y,z) = e^{ik_x x} e^{ik_z z} \phi(y)$$

This form is suggested as $\mathcal{H}$ is translationally invariant in x and z, but not in y (due to our particular choice for $\vec{A}$)

Substitute this ψ into above Schrodinger Equation to get

$$\frac{1}{2m}\left[\hbar^2 k_z^2 - \hbar^2\frac{\partial^2}{\partial y^2} + \left(\hbar k_x - \frac{e}{c} H y\right)^2\right]\phi(y) = \varepsilon\phi(y)$$

or

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial y^2} + \frac{1}{2m}\left(\hbar k_x - \frac{e}{c} H y\right)^2\phi = \left(\varepsilon - \frac{\hbar^2 k_z^2}{2m}\right)\phi$$

define y_0 such that $\hbar k_x \equiv \frac{eH}{c} y_0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial y^2} + \frac{1}{2m}\left(\frac{eH}{c}\right)^2 (y - y_0)^2\phi = \left(\varepsilon - \frac{\hbar^2 k_z^2}{2m}\right)\phi$$

define cyclotron frequency $\omega_c \equiv \frac{eH}{mc}$

(a classical charged particle in uniform $\vec{H}$ moves in a circular orbit with angular velocity ω_c)

Finally we get

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2} m\omega_c^2 (y-y_0)^2\right]\phi(y) = \left(\varepsilon - \frac{\hbar^2 k_z^2}{2m}\right)\phi(y)$$

This is just the Hamiltonian for a simple harmonic oscillator of frequency ω_c that is centered at $y = y_0$.

We know the eigen values of energy of the harmonic oscillator are just

$$\hbar\omega_c(n+1/2) \qquad n = 0, 1, 2, \ldots$$

so we then have

$$\varepsilon - \frac{\hbar^2 k_z^2}{2m} = \hbar\omega_c(n+1/2)$$

or the energy eigenvalues of our particle are

$$\varepsilon = \underbrace{\frac{\hbar^2 k_z^2}{2m}}_{\uparrow} + \underbrace{\hbar\omega_c(n+1/2)}_{\uparrow}$$

kinetic energy of motion along $\hat{z}$ parallel to $\vec{H}$ — kinetic energy of orbital motion in xy plane $\perp$ to $\vec{H}$

The wave functions $\phi_n(y)$ are the usual harmonic oscillator wavefunctions (gaussian × Hermite polynomial) only centered at y_0.

We can therefore write our solution in terms of 3 quantum numbers, k_x, k_z, n

$$\begin{cases} \psi_{k_x,k_y,n}(x,y,z) = e^{ik_x x} e^{ik_z z} \tilde{\Phi}_n(y-y_0) \\ \qquad (\tilde{\Phi} \text{ is h.o. wavefunction centered at origin}) \\ \mathcal{E}(k_x,k_z,n) = \dfrac{\hbar^2 k_z^2}{2m} + \hbar\omega_c(n+1/2) \end{cases}$$

where $y_0 = \hbar k_x \dfrac{c}{eH} = \dfrac{\hbar k_x}{m\omega_c}$

Note $\mathcal{E}$ is independent of k_x so for fixed k_z and n there are many degenerate states corresponding to the different possible choices for k_x.

What are the possible values of k_x?

If we take periodic boundary conditions along x, $\psi(x+L_x,y,z) = \psi(x,y,z)$ then we must have

$$e^{ik_x x} = e^{ik_x(x+L_x)} \Rightarrow k_x = \frac{2\pi}{L_x} \times (\text{integer})$$

But k_x also determines the value of y_0 about which the wavefunction is centered in the y-direction. Therefore we must have

$$0 \le y_0 \le L_y \Rightarrow 0 \le \frac{\hbar k_x}{m\omega_c} \le L_y$$

$$\Rightarrow k_{x\,max} = L_y \frac{m\omega_c}{\hbar} = \frac{L_y}{\hbar}\frac{m\,eH}{mc} = \frac{L_y eH}{\hbar c}$$

Combining these two conditions we have the the number of allowed values k_x can take is given by

$$\frac{k_{x\,max}}{\Delta k_x} = \frac{L_y \frac{eH}{\hbar c}}{\left(\frac{2\pi}{L_x}\right)} = L_x L_y \frac{eH}{2\pi \hbar c} = L_x L_y \frac{eH}{hc}$$

$$= \frac{L_x L_y H}{\left(\frac{hc}{e}\right)}$$

to get the number of allowed electron states with energy $\frac{\hbar k_z^2}{2m} + \hbar\omega_c(n+\frac{1}{2})$ we should multiply above by a factor of 2 for the two possible spin states.

$$\text{Degeneracy } W = \frac{2 L_x L_y H}{\left(\frac{hc}{e}\right)} = \frac{\Phi}{\left(\frac{hc}{2e}\right)} = \frac{\Phi}{\Phi_0}$$

Where $\Phi = L_x L_y H$ is the total magnetic flux penetrating the system, and

$\Phi_0 = \frac{hc}{2e}$ has units of magnetic flux and is called the "flux quantum"

$\Phi_0 = 2.07 \times 10^{-7}$ gauss-cm^2

degeneracy is $\frac{\Phi}{\Phi_0}$ = number of flux quanta

Consider now just the motion of the electron in the xy plane. The energy of this motion is

$$\tilde{\varepsilon} = \varepsilon - \frac{\hbar^2 k_z^2}{2m} = \hbar\omega_c(n+1/2) \qquad n = 0, 1, 2, \ldots$$

The states corresponding to a given value of n are called the "n^{th} Landau level". The n^{th} Landau level has a degeneracy of Φ/Φ_0, or equivalently, the number of electrons per unit area that one can put into a given Landau level is

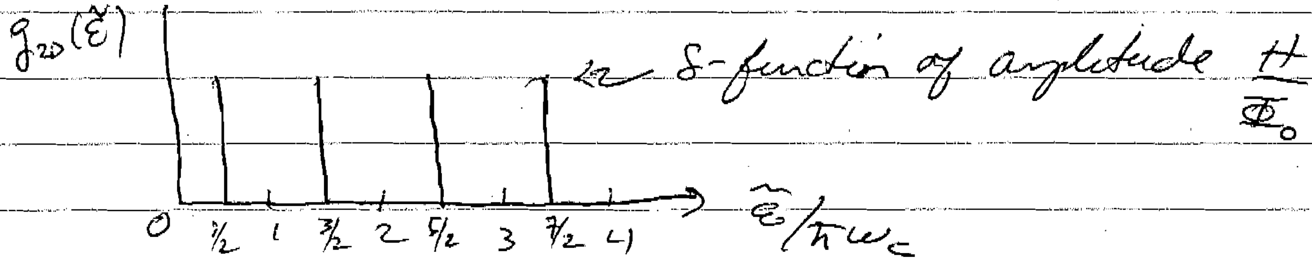
$$\frac{1}{L_x L_y}\frac{\Phi}{\Phi_0} = \frac{H}{\Phi_0}$$

We can summarize this by giving the density of states for the energy $\tilde{\varepsilon}$ in the xy plane

$g_{2D}(\tilde{\varepsilon})\,\Delta\varepsilon$ = number of electron states per unit area with energy in the range $\tilde{\varepsilon}$ to $\tilde{\varepsilon}+\Delta\varepsilon$.

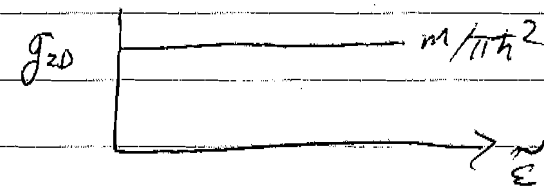
Since there are only states at the discrete energy values $\hbar\omega_c(n+1/2)$, $g_{2D}(\tilde{\varepsilon})$ is a sum of δ-functions at these discrete values - the amplitude of each δ-function is just the degeneracy per area H/Φ_0

$$g_{2D}(\tilde{\varepsilon}) = \sum_n \frac{H}{\Phi_0}\,\delta(\tilde{\varepsilon} - \hbar\omega_c(n+1/2))$$



We can compare this to the 2D density of states when $H=0$. From problem (3b) of HW set 1 you will find that at $H=0$, $g_{2D}(\tilde{\varepsilon})$ is a constant

$$H=0: \quad g_{2D}(\tilde{\varepsilon}) = \frac{m}{\pi\hbar^2}$$



To compare $H=0$ with $H>0$, consider computing the average density of state for $H>0$ where we average over an energy interval large compared to the spacing between the Landau levels $\hbar\omega_c$.

$$\text{average density of states } \bar{g} = \frac{(\#\ \delta\text{-function spikes in } \Delta\mathcal{E})}{\text{interval width } \Delta\mathcal{E}} \times \frac{H}{\Phi_0}$$

If we take $\Delta\mathcal{E} = M\hbar\omega_c$ for a large integer M, then on average there will be M δ-function spikes in this interval, so

$$\bar{g} = \frac{M \times \frac{H}{\Phi_0}}{M\hbar\omega_c} = \frac{H}{\left(\frac{hc}{2e}\right)} \frac{1}{\hbar\left(\frac{eH}{mc}\right)} = \frac{m}{\pi\hbar^2}$$

So average density of state at $H>0$

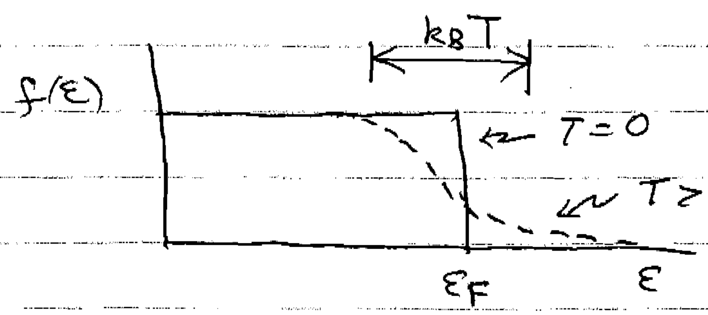
$$\bar{g} = \frac{m}{\pi\hbar^2} = \text{constant density of states at } H=0$$

So turning on the magnetic field bunches the energy eigenstates up into discrete levels, but the average number of states per unit energy remains the same (provided we average on interval $\gg \hbar\omega_c$)

Suppose we had an actual 2D electron gas.
One can think of making this in a thin metallic film or a semiconductor inversion layer where the gas is confined to a region in space along $\hat{z}$ so small that only the lowest allowed value of k_z is occupied, ie $\frac{2\pi}{L_z} = \Delta k_z$ gives $\frac{\hbar^2(\Delta k_z)^2}{2m}$ larger than all other energy scales.

What is necessary so that one could detect the difference between the discrete Landau level structure at finite $H > 0$, and the average density of states which is equal to its $H=0$ value?

If f is the Fermi function, we know that finite temperature smears out the sharp cutoff at $\varepsilon = \varepsilon_F$ that exists at $T=0$.



To see the Landau level structure we thus need this smearing to be small on the scale of the spacing between the Landau levels

ie. need $k_BT \ll \hbar\omega_c$

using $\omega_c = \frac{eH}{mc}$ and m the free electron mass one can compute

$$\omega_c = 1.76 \times 10^{11}\ \text{sec}^{-1} \quad \text{for a } H = 1 \text{ tesla} = 10^4 \text{ gauss}$$

magnetic field.

1 tesla is a big field. In a laboratory setup such as in BL one can buy a 10 tesla magnet. Larger field strengths require specialized facilities

So for $H = 1$ tesla, $\boxed{\frac{\hbar\omega_c}{k_B} = 1.34\ ^\circ K}$

So in a 1 tesla field one needs to go well below 1°K to see Landau level structure. In a 10 tesla field one needs to go well below 10°K. So quite low temperatures are needed.

There is a second condition. In solving Schrodinger's equation for the Landau levels, we ignored any sources of electron scattering (scattering off phonons, plasmons, lattice impurities, etc.) If τ is the scattering time, including such scattering generally leads, via the uncertainty principle, to a broading of the energy levels of the eigenstates to a finite width $\delta E \sim \frac{\hbar}{\tau}$

So to see Landau level structure we need

$$\delta\varepsilon \ll \hbar\omega_c \Rightarrow \frac{\hbar}{\tau} \ll \hbar\omega_c$$

$$\Rightarrow \omega_c\tau \gg 1$$

using $\omega_c = 1.76 \times 10^{11}\ \text{sec}^{-1}$ in $H = 1$ tesla and from resistivity measurements used to estimate τ from Drude's model we get

room temp $\tau \sim 10^{-14}$ sec , $\omega_c\tau \sim 0.00176$
77°K (liquid N) $\tau \sim 10^{-13}$ sec , $\omega_c\tau \sim 0.0176$

we again see that we will need very low temperatures (large τ) to get $\omega_c\tau \gg 1$.

Landau level structure is typically only observable if one goes down to liquid Helium temperatures ~ 5°K.