3D density of states

$$\varepsilon = \tilde{\varepsilon} + \frac{\hbar^2 k_z^2}{2m}$$

to compute $g(\varepsilon)\Delta\varepsilon$ we need to sum over all values of $\tilde{\varepsilon}$ and k_z such that

$$\varepsilon \le \tilde{\varepsilon} + \frac{\hbar^2 k_z^2}{2m} \le \varepsilon + \Delta\varepsilon$$

weighting this sum by $g_{2D}(\tilde{\varepsilon})$ the density of states at $\tilde{\varepsilon}$

$$g(\varepsilon)\Delta\varepsilon = \int_0^{\varepsilon} d\tilde{\varepsilon}\, g_{2D}(\tilde{\varepsilon}) \left[\int_{k_0}^{k_0+\Delta k} \frac{dk_z}{2\pi} + \int_{-(k_0+\Delta k)}^{-k_0} \frac{dk_z}{2\pi} \right] \quad (1)$$

$$+ \int_{\varepsilon}^{\varepsilon+\Delta\varepsilon} d\tilde{\varepsilon}\, g_{2D}(\tilde{\varepsilon}) \left[\int_0^{k_0+\Delta k} \frac{dk_z}{2\pi} + \int_{-(k_0+\Delta k)}^{0} \frac{dk_z}{2\pi} \right] \quad (2)$$

where

$$k_0 = \sqrt{\frac{2m}{\hbar^2}(\varepsilon - \tilde{\varepsilon})}$$

$$k_0 + \Delta k = \sqrt{\frac{2m}{\hbar^2}(\varepsilon + \Delta\varepsilon - \tilde{\varepsilon})}$$

$$\Rightarrow \Delta k = \frac{1}{2}\sqrt{\frac{2m}{\hbar^2}} \frac{\Delta\varepsilon}{\sqrt{\varepsilon - \tilde{\varepsilon}}}$$

$$\text{term (2)} = \int_{\varepsilon}^{\varepsilon+\Delta\varepsilon} d\tilde{\varepsilon}\, g_{2D}(\tilde{\varepsilon}) \frac{2}{2\pi}\sqrt{\frac{2m}{\hbar^2}(\varepsilon + \Delta\varepsilon - \tilde{\varepsilon})}$$

over range of integration, since $\Delta\varepsilon$ small, approx $g_{2D}(\tilde{\varepsilon}) \simeq g_{2D}(\varepsilon)$

$$\simeq \frac{2}{2\pi}\sqrt{\frac{2m}{\hbar^2}}\, g_{2D}(\varepsilon) \int_{\varepsilon}^{\varepsilon+\Delta\varepsilon} d\tilde{\varepsilon}\, \sqrt{\varepsilon + \Delta\varepsilon - \tilde{\varepsilon}}$$

$$= \frac{2}{2\pi}\sqrt{\frac{2m}{\hbar^2}}\, g_{2D}(\tilde{\varepsilon})\left[-\frac{2}{3}\left(\varepsilon+\Delta\varepsilon-\tilde{\varepsilon}\right)^{3/2}\right]_{\varepsilon}^{\varepsilon+\Delta\varepsilon}$$

$$= \frac{4}{3\pi}\sqrt{\frac{2m}{\hbar^2}}\, g_{2D}(\tilde{\varepsilon})\,(\Delta\varepsilon)^{3/2}$$

$\Rightarrow$ this term gives a vanishing contribution to $g(\varepsilon)\Delta\varepsilon$

since $\frac{(\Delta\varepsilon)^{3/2}}{\Delta\varepsilon} = \sqrt{\Delta\varepsilon} \to 0$ as $\Delta\varepsilon \to 0$

So we only need the term ①

$$g(\varepsilon)\Delta\varepsilon = \int_0^{\varepsilon} d\tilde{\varepsilon}\, g_{2D}(\tilde{\varepsilon})\; 2\,\frac{\Delta k}{2\pi}$$

$$= \int_0^{\varepsilon} d\tilde{\varepsilon}\, g_{2D}(\tilde{\varepsilon})\,\frac{2}{2\pi}\,\frac{\Delta\varepsilon}{\sqrt{\varepsilon-\tilde{\varepsilon}}}\sqrt{\frac{2m}{\hbar^2}}\,\frac{1}{2}$$

$$\boxed{g(\varepsilon) = \frac{1}{2\pi}\sqrt{\frac{2m}{\hbar^2}}\int_0^{\varepsilon} d\tilde{\varepsilon}\,\frac{g_{2D}(\tilde{\varepsilon})}{\sqrt{\varepsilon-\tilde{\varepsilon}}}}$$

for $H=0$ $\quad g_{2D}(\tilde{\varepsilon}) = \frac{m}{\pi\hbar^2}$ constant

$$g(\varepsilon) = \frac{1}{2\pi}\sqrt{\frac{2m}{\hbar^2}}\,\frac{m}{\pi\hbar^2}\int_0^{\varepsilon} d\tilde{\varepsilon}\,\frac{1}{\sqrt{\varepsilon-\tilde{\varepsilon}}}$$

$$= \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left[-2\sqrt{\varepsilon-\tilde{\varepsilon}}\right]_0^{\varepsilon}$$

$$g(\varepsilon) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}$$ $\leftarrow$ this is the correct 3D density of states!

For $H>0$ we now need to use $g_{2D}(\tilde{\varepsilon})$ that describes the Landau levels

$$g(\varepsilon) = \frac{1}{2\pi}\sqrt{\frac{2m}{\hbar^2}}\int_0^{\varepsilon} d\tilde{\varepsilon}\sum_n \delta(\tilde{\varepsilon}-\hbar\omega_c(n+1/2))\frac{H}{\Phi_0}\frac{1}{\sqrt{\varepsilon-\tilde{\varepsilon}}}$$

use $\Phi_0 = \dfrac{hc}{2e} = \dfrac{\pi\hbar c}{e}$, $\omega_c = \dfrac{eH}{mc} \Rightarrow H = \dfrac{mc\omega_c}{e}$

$$g(\varepsilon) = \frac{1}{2\pi}\sqrt{\frac{2m}{\hbar^2}}\;\frac{mc\omega_c}{e}\;\frac{e}{\pi\hbar c}\sum_n{}' \frac{1}{\sqrt{\varepsilon-\hbar\omega_c(n+1/2)}}$$

sum over n such that $\varepsilon \geq \hbar\omega_c(n+1/2)$

$$g(\varepsilon) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\hbar\omega_c\sum_n{}'\frac{1}{\sqrt{\varepsilon-\hbar\omega_c(n+1/2)}}$$

let $x \equiv \varepsilon/\hbar\omega_c$

$$g(\varepsilon) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\hbar\omega_c}\sum_n{}'\frac{1}{\sqrt{x-n-1/2}}$$

sum on n such that $x \geq n+1/2$

Compare to $H=0$

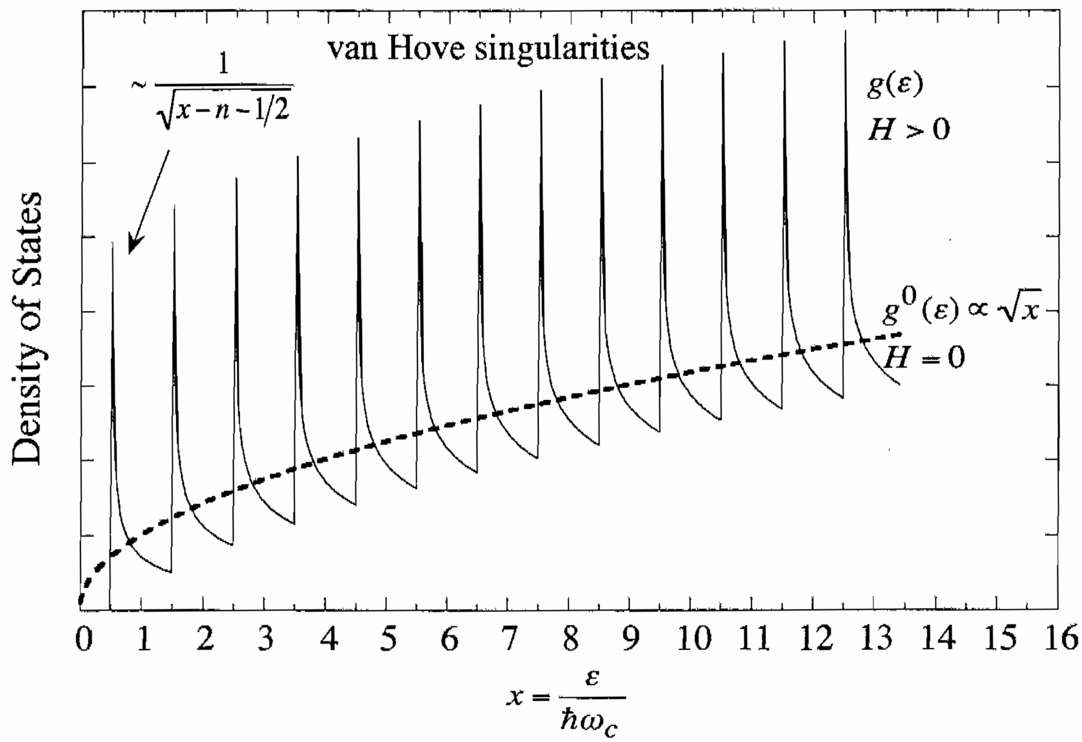
$$g^0(\varepsilon) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon} = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\hbar\omega_c}\sqrt{x}$$

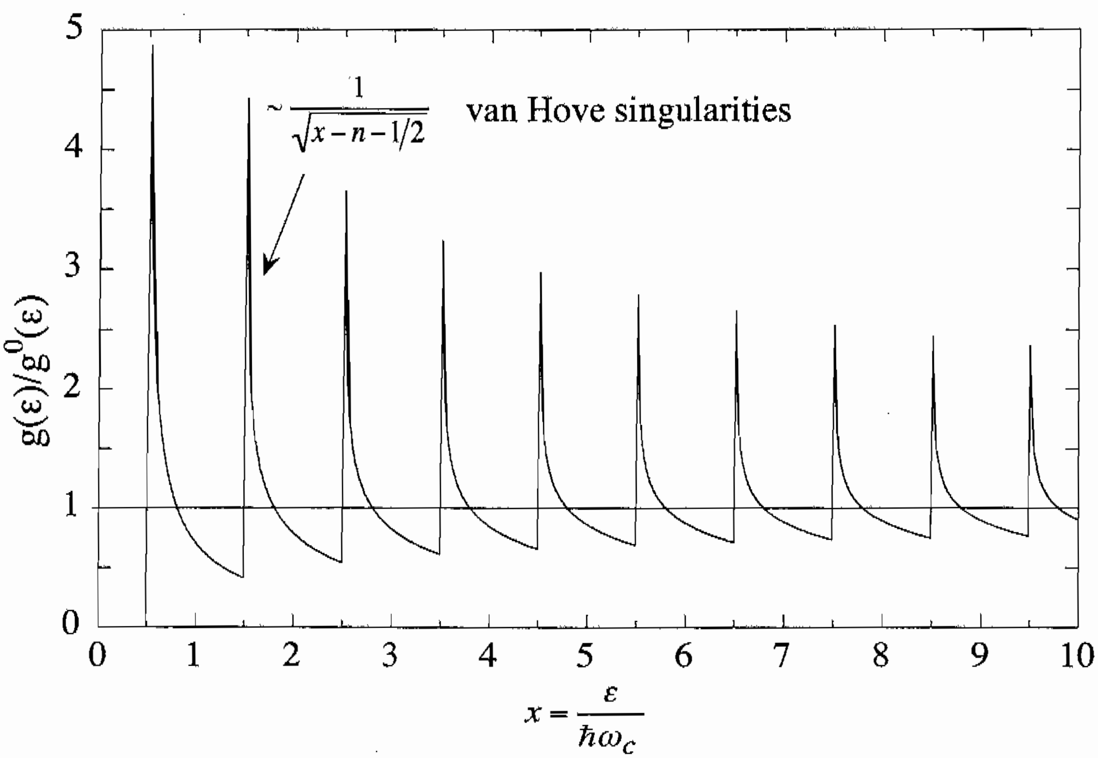
$$= A\sqrt{x}$$

We get $\quad g(\varepsilon) = \frac{1}{2}A\sum_n{}'\frac{1}{\sqrt{x-n-1/2}}$

$$\boxed{\frac{g(\varepsilon)}{g^0(\varepsilon)} = \frac{1}{2\sqrt{x}}\sum_n{}'\frac{1}{\sqrt{x-n-1/2}} \qquad x = \frac{\varepsilon}{\hbar\omega_c}}$$

n such that $x \geq n+1/2$





$g(\varepsilon)$ oscilates with period of $\hbar\omega_c$
with divergent but integrable
"van Hove" singularites at $\varepsilon_n = \hbar\omega_c(n+1/2)$

Having found the density of states $g(\varepsilon)$ for electrons in a magnetic field H, our goal now is to compute the magnetic susceptibility χ. We will do this by computing the total energy E.

Just like the energy of an electron $\overset{\text{intrinsic magnetic moment}}{\vec{\mu}}$ in a magnetic field $\vec{H}$ is $-\vec{\mu}\cdot\vec{H}$, the energy of a magnetization density $\vec{M}$ in a magnetic field $\vec{H}$ is $-V\vec{M}\cdot\vec{H}$, where V is the volume of the system.

Thus $E(H) = E(0) - V\vec{M}\cdot\vec{H}$ for small $\vec{H}$

and $\vec{M} = \frac{-1}{V}\frac{\partial E}{\partial \vec{H}}$

The magnetic susceptibility is then defined as $\chi \equiv \left.\frac{\partial M}{\partial H}\right|_{H=0}$ and so

$\chi = -\frac{1}{V}\left.\frac{\partial^2 E}{\partial H^2}\right|_{H=0}$. So in the limit of small H we have

$$E(H) = E(0) - \frac{1}{2}V\chi H^2$$

So from the quadratic dependence of $E(H)$ on H at small H, we can extract χ.

To compute $E(H)$ we need two steps

① compute how the Fermi energy ε_F varies with the applied field H.

② Knowing ε_F, compute $\int_0^{\varepsilon_F} d\varepsilon'\, g(\varepsilon')\,\varepsilon'$ to get E

Next we want to compute the total ground state energy E. From the dependence of E on H we will get the Landau diamagnetic susceptibility.

First we introduce the "integrated density of states"

$$G(\varepsilon) = \int_0^{\varepsilon} d\varepsilon' \, g(\varepsilon')$$

From the condition $n = G(\varepsilon_F)$ we will determine the Fermi energy ε_F.

For $H = 0$

$$g^0(\varepsilon) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}$$

$$G^0(\varepsilon) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{3}\varepsilon^{3/2}$$

$n = G^0(\varepsilon_F^0)$ $\quad\varepsilon_F^0$ is Fermi energy when $H = 0$

$$\Rightarrow \quad \varepsilon_F^0 = \left[\frac{3}{2} n \, 2\pi^2 \left(\frac{\hbar^2}{2m}\right)^{3/2}\right]^{2/3} = \frac{\hbar^2}{2m}\underbrace{\left(3\pi^2 n\right)^{2/3}}_{=k_F}$$

we also can write $\frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} = \frac{3}{2} n \left(\varepsilon_F^0\right)^{-3/2}$

So

$$\boxed{\begin{aligned} g^0(\varepsilon) &= \frac{3}{2}\frac{n}{\varepsilon_F^0}\sqrt{\frac{\varepsilon}{\varepsilon_F^0}} \\ G^0(\varepsilon) &= n\left(\frac{\varepsilon}{\varepsilon_F^0}\right)^{3/2} \end{aligned}}$$

For $H > 0$

$$g(\varepsilon) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\hbar\omega_c}\ \frac{1}{2}\sum_n{}' \frac{1}{\sqrt{x-n-1/2}}$$

$$= \frac{3}{2}\frac{m}{\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\ \frac{1}{2}\sum_n{}' \frac{1}{\sqrt{x-n-1/2}}$$

where $x = \varepsilon/\hbar\omega_c$ and sum is over integer n such that $x \geq n+1/2$

$$G(\varepsilon) = \int_0^{\varepsilon} d\varepsilon'\, g(\varepsilon') = \hbar\omega_c \int_0^{x} dx'\, g(x')$$

$$= \frac{3}{2}\frac{m}{\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\ \hbar\omega_c \int_0^{x} dx'\ \frac{1}{2}\sum_n{}' \frac{1}{\sqrt{x'-n-1/2}}$$

A given term n in the sum only appears for those values of x' such that $x' \geq n+1/2$. Hence we can rewrite the sum and the integrals as

$$G(\varepsilon) = \frac{3}{2}\frac{m}{\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\ \hbar\omega_c \sum_{n=0}^{n_{max}} \frac{1}{2}\int_{n+1/2}^{x} dx'\ \frac{1}{\sqrt{x'-n-1/2}}$$

n_{max} is largest integer such that $x \geq n_{max}+1/2$

[lower limit becomes $n+1/2$ since the "n" term is only present in the sum when $x' \geq n+1/2$

$$G(\varepsilon) = \frac{3}{2}\frac{m}{\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\ \hbar\omega_c \sum_{n=0}^{n_{max}} \left[\sqrt{x'-n-1/2}\right]_{n+1/2}^{x}$$

$$G(\varepsilon) = \frac{3}{2}\frac{m}{\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\ \hbar\omega_c \sum_{n=0}^{n_{max}} \sqrt{x-n-1/2}$$

Compare to $G^0(\varepsilon) = n\left(\frac{\varepsilon}{\varepsilon_F^0}\right)^{3/2}$ and we have

$$\boxed{\begin{array}{l} \dfrac{G(\varepsilon)}{G^0(\varepsilon)} = \dfrac{3}{2}\dfrac{1}{x^{3/2}}\displaystyle\sum_{n=0}^{n_{max}}\sqrt{x-n-1/2} \\ \\ x = \varepsilon/\hbar\omega_c \qquad\qquad n_{max}+\tfrac{1}{2} \le x \end{array}}$$

define $f(x) \equiv \frac{3}{2}\frac{1}{x^{3/2}}\sum_{n=0}^{n_{max}}\sqrt{x-n-1/2}$

$$G(\varepsilon) = G^0(\varepsilon) f(x)$$

We can use this to compute the Fermi energy ε_F when $H>0$. We will see that ε_F must shift from its $H=0$ value as H is turned on

Let $\varepsilon_F = \varepsilon_F^0 + \delta\varepsilon$

Fermi energy at finite H — Fermi energy when $H=0$ — shift in Fermi energy when H turned on

determines ε_F for fixed density n

$$n = G(\varepsilon_F) = G^0(\varepsilon_F^0+\delta\varepsilon) f(x^0+\delta x), \quad \begin{cases} x^0 = \dfrac{\varepsilon_F^0}{\hbar\omega_c} \\ \delta x = \dfrac{\delta\varepsilon}{\hbar\omega_c} \end{cases}$$

For small H we expect $\delta\varepsilon$ is small so expand to lowest order in $\delta\varepsilon$

$$n = \left[G^0(\varepsilon_F^0) + \frac{dG^0(\varepsilon_F^0)}{d\varepsilon}\delta\varepsilon\right]\left[f(x^0) + f'(x^0)\frac{\delta\varepsilon}{\hbar\omega_c}\right]$$

now $\dfrac{dG^0(\varepsilon_F^0)}{d\varepsilon} = g^0(\varepsilon_F^0) = \dfrac{3}{2}\dfrac{n}{\varepsilon_F^0}$

and also $n = G^0(\varepsilon_F^0)$

So

$$n = \left[n + \frac{3}{2}\frac{n}{\varepsilon_F^0}\,\delta\varepsilon \right]\left[f(x^0) + f'(x^0)\frac{\delta\varepsilon}{\hbar\omega_c} \right]$$

to linear order in $\delta\varepsilon$

$$n = n f(x^0) + \left[\frac{3}{2}\frac{n}{\varepsilon_F^0} f(x^0) + \frac{n f'(x^0)}{\hbar\omega_c} \right]\delta\varepsilon$$

$$1 = f(x^0) + \left[\frac{3}{2}\frac{\hbar\omega_c}{\varepsilon_F^0} f(x^0) + f'(x^0) \right]\frac{\delta\varepsilon}{\hbar\omega_c}$$

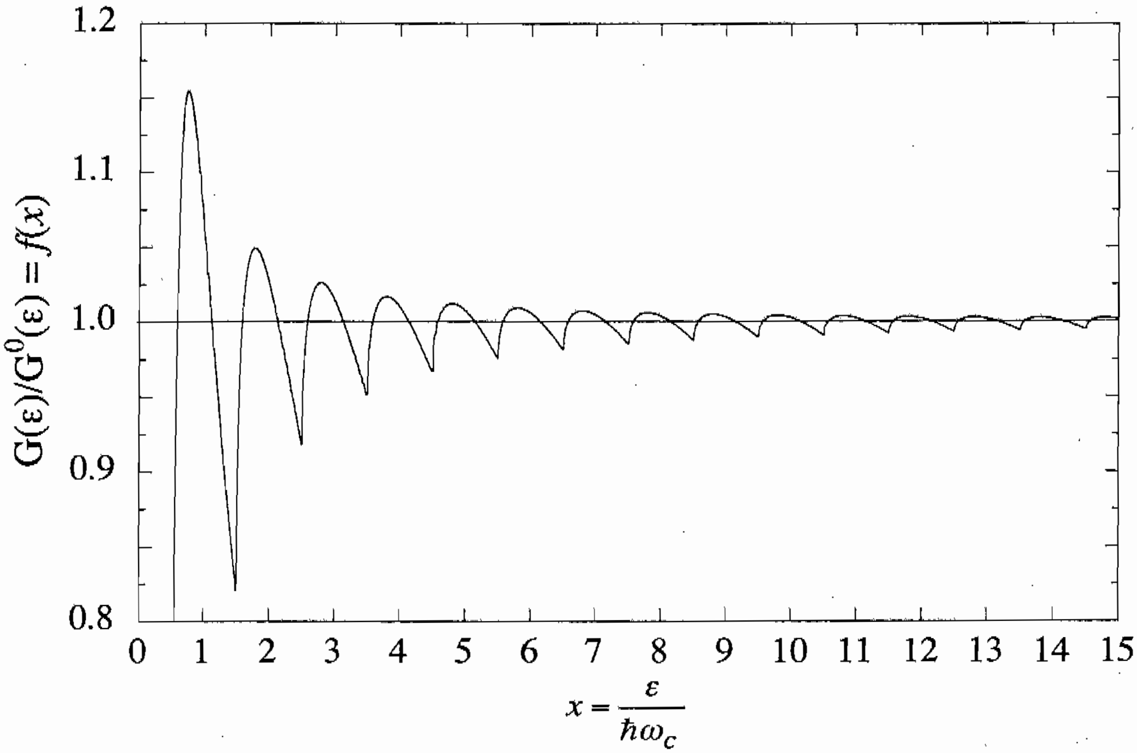
shift in Fermi energy due to finite H

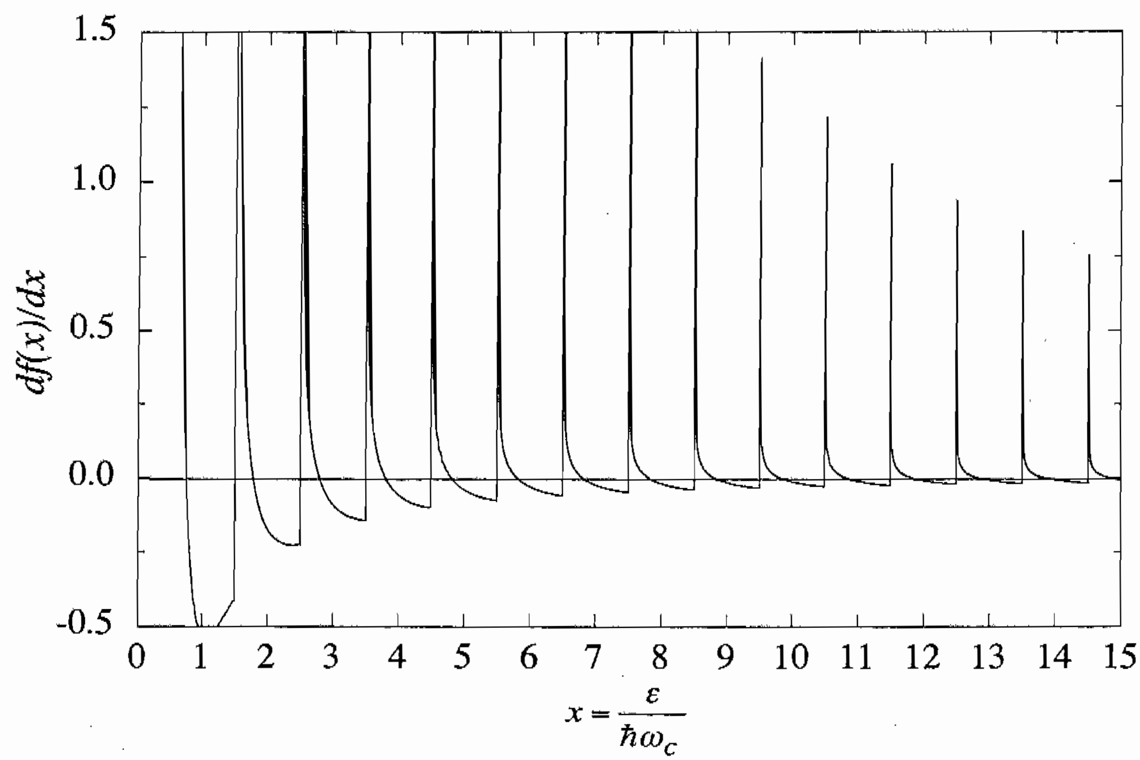
$$\boxed{\frac{\delta\varepsilon}{\hbar\omega_c} = \delta x = \frac{1 - f(x^0)}{f'(x^0) + \frac{3}{2}\frac{f(x^0)}{x^0}}}$$

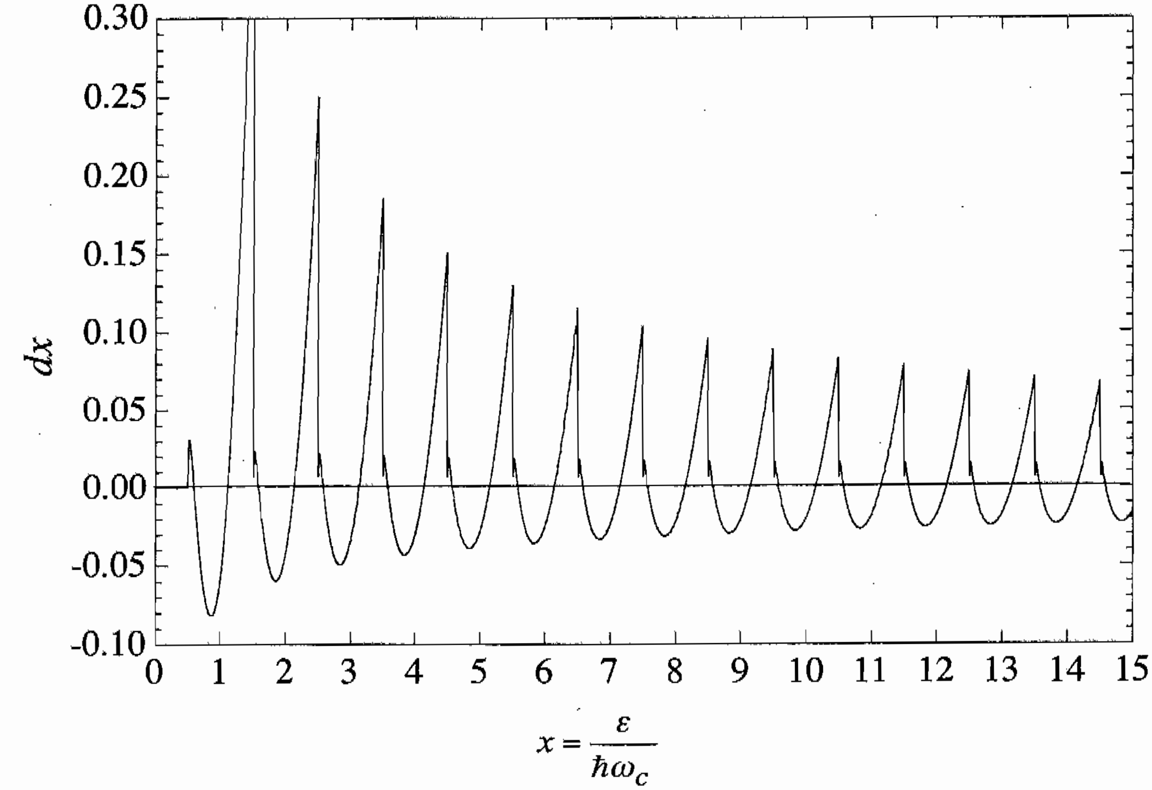
for small H, $x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c} \to \infty$, so second term in denominator gets small. But we need to keep it because at some x^0 we can have $f'(x^0) = 0$ and the second term is then needed to keep δx from diverging.

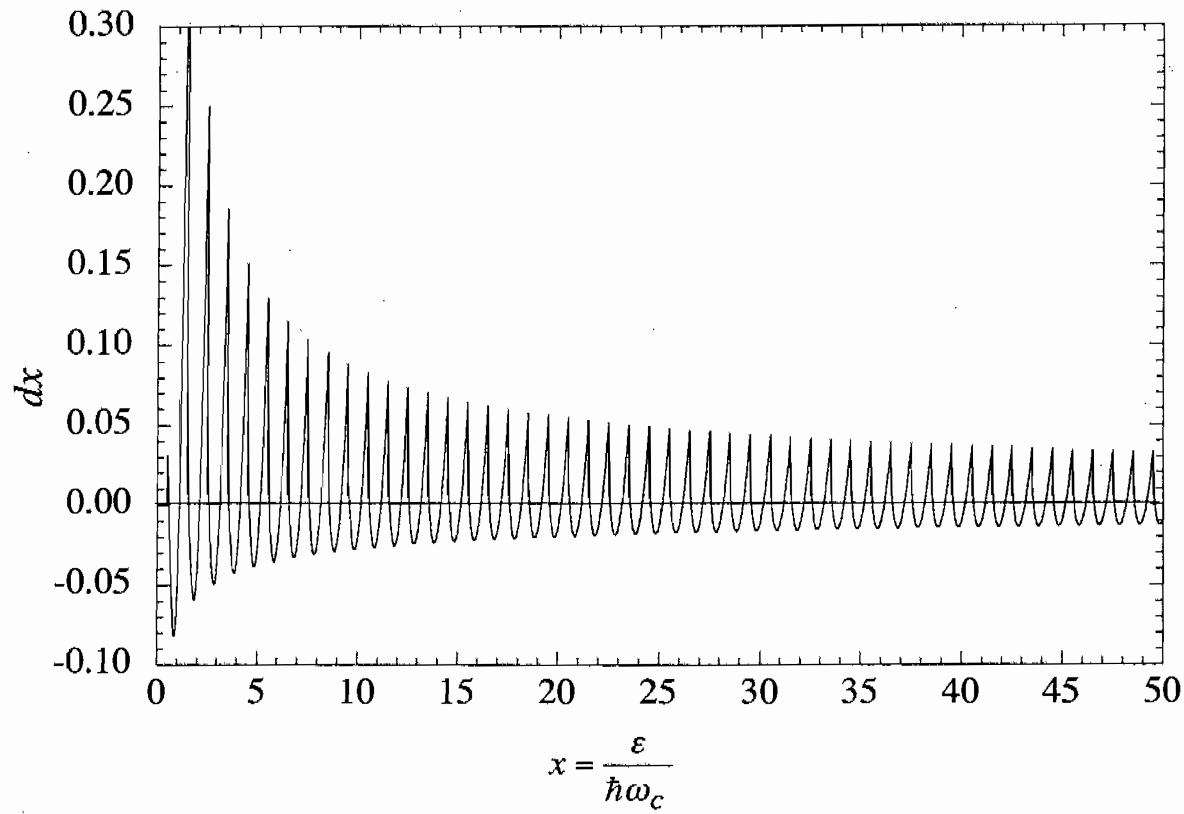
For large x^0 we find δx is indeed small, $\delta x \ll 1$, as desired.

Having found the Fermi energy $\varepsilon_F = \varepsilon_F^0 + \delta\varepsilon$ at finite H we can now proceed to compute the ground state energy!


1.2
1.1
1.0
0.9
0.8
G(ε)/G⁰(ε) = f(x)
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$x = \frac{\varepsilon}{\hbar\omega_c}$


1.5
1.0
0.5
0.0
-0.5
df(x)/dx
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$x = \frac{\varepsilon}{\hbar\omega_c}$


0.30
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
-0.10
dx
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$x = \frac{\varepsilon}{\hbar\omega_c}$


0.30
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
-0.10
dx
0 5 10 15 20 25 30 35 40 45 50
$x = \frac{\varepsilon}{\hbar\omega_c}$