Energy. Let us define

$$\frac{E(\varepsilon)}{V} = \int_0^{\varepsilon} d\varepsilon' \, g(\varepsilon')\varepsilon'$$

Then the total ground state energy will be $E(\varepsilon_F)$

$\underline{H=0}$ $\quad g^0(\varepsilon) = \frac{3}{2} \frac{n}{\varepsilon_F^0} \sqrt{\frac{\varepsilon}{\varepsilon_F^0}}$

$$\frac{E^0(\varepsilon)}{V} = \frac{3}{2} \frac{n}{(\varepsilon_F^0)^{3/2}} \int_0^{\varepsilon} d\varepsilon' \, \varepsilon'^{3/2}$$

$$= \frac{3}{2} \cdot \frac{2}{5} \frac{n}{(\varepsilon_F^0)^{3/2}} \varepsilon^{5/2}$$

$$\boxed{\frac{E^0(\varepsilon)}{V} = \frac{3}{5} n \left(\frac{\varepsilon}{\varepsilon_F^0}\right)^{3/2} \varepsilon}$$

from above we get familiar result

$$\frac{E^0(\varepsilon_F^0)}{V} = \frac{3}{5} n \varepsilon_F^0 \quad \text{at } H=0$$

$\underline{H>0}$

$$g(\varepsilon) = \frac{3}{2} \frac{n}{\varepsilon_F^0} \sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}} \, \frac{1}{2} \sum_n{}' \frac{1}{\sqrt{x-n-1/2}}$$

term n appears in sum only if $x \geq n + 1/2$

$$\frac{E(\varepsilon)}{V} = \int_0^{\varepsilon'} d\varepsilon' g(\varepsilon')\varepsilon' = (\hbar\omega_c)^2 \int_0^x dx' \, g(x')\, x'$$

$$= \frac{3}{2} \frac{n}{\varepsilon_F^0} \sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}} (\hbar\omega_c)^2 \sum_n{}' \frac{1}{2} \int_0^x dx' \frac{x'}{\sqrt{x'-n-1/2}}$$

as we explained in discussion concerning the computation of $G(\varepsilon)$, we can rewrite the sum and integrals as

$$\frac{E(\varepsilon)}{V} = \frac{3}{2}\frac{m}{\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\,(\hbar\omega_c)^2\sum_{n=0}^{n_{max}}\frac{1}{2}\int_{n+1/2}^{x} dx'\,\frac{x'}{\sqrt{x'-n-1/2}}$$

where n_{max} is largest integer such that $x \geq n_{max}+\frac{1}{2}$

We can look up the integral in a table to find

$$\int dx\,\frac{x}{\sqrt{x-a}} = \frac{2}{3}(x+2a)\sqrt{x-a}$$

So

$$\frac{E(\varepsilon)}{V} = \frac{m}{2\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\,(\hbar\omega_c)^2\sum_{n=0}^{n_{max}}\Big[(x'+2n+1)\sqrt{x'-n-1/2}\Big]_{n+1/2}^{x}$$

$$\frac{E(\varepsilon)}{V} = \frac{m}{2\varepsilon_F^0}\sqrt{\frac{\hbar\omega_c}{\varepsilon_F^0}}\,(\hbar\omega_c)^2\sum_{n=0}^{n_{max}}(x+2n+1)\sqrt{x-n-1/2}$$

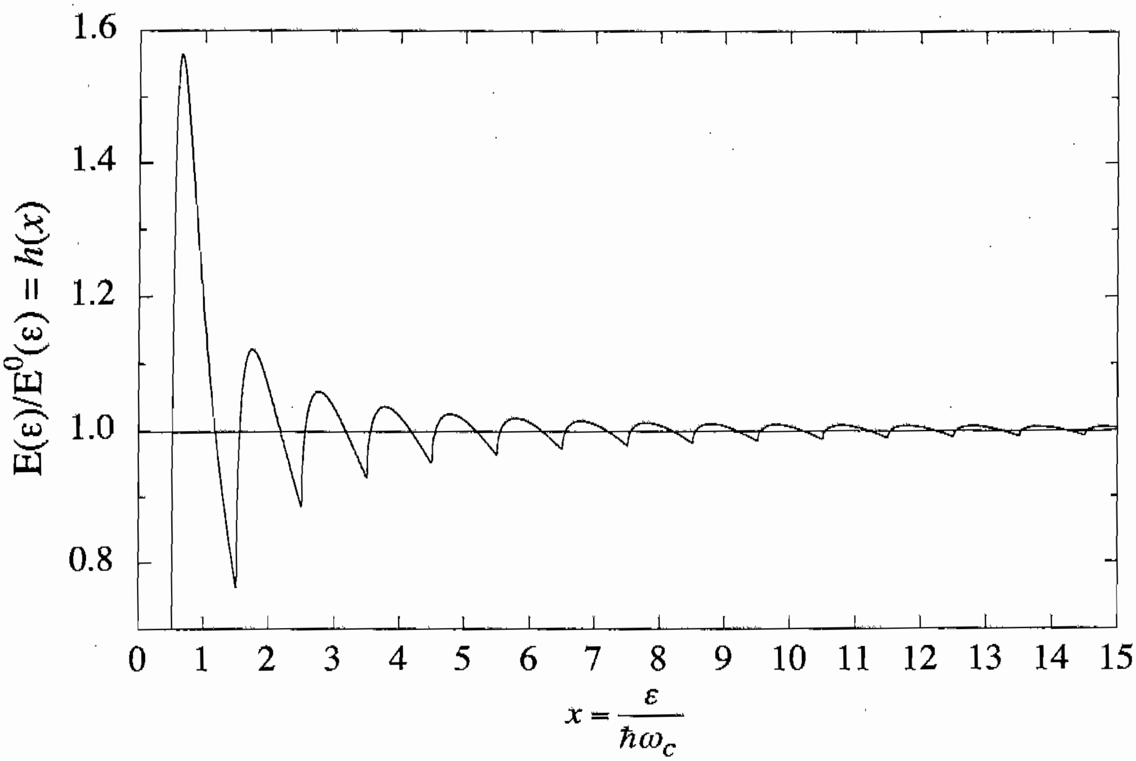
Compare to $\dfrac{E^0(\varepsilon)}{V} = \dfrac{3}{5}m\left(\dfrac{\varepsilon}{\varepsilon_F^0}\right)^{3/2}\varepsilon$ and we have

$$\boxed{\frac{E(\varepsilon)}{E^0(\varepsilon)} = \frac{5}{6}\frac{1}{x^{5/2}}\sum_{n=0}^{n_{max}}(x+2n+1)\sqrt{x-n-1/2}}$$

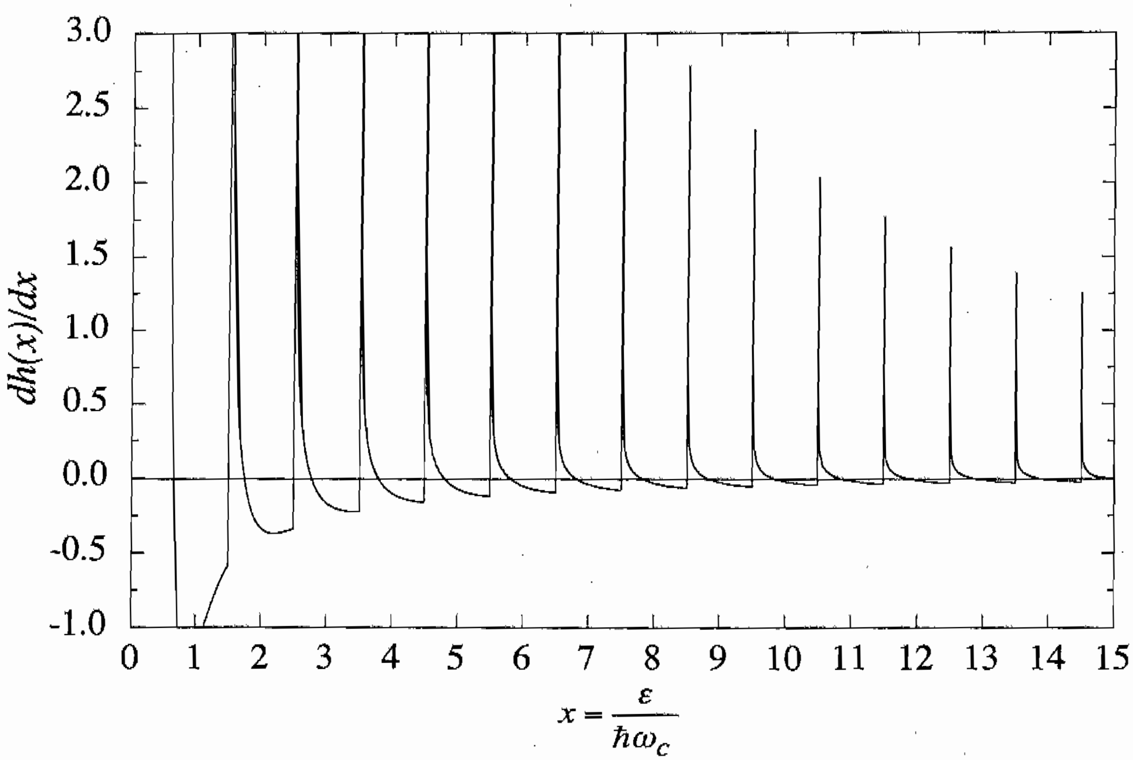
$x = \varepsilon/\hbar\omega_c$ and n_{max} such that $x \geq n_{max}+1/2$

define $h(x) \equiv \dfrac{5}{6}\dfrac{1}{x^{5/2}}\displaystyle\sum_{n=0}^{n_{max}}(x+2n+1)\sqrt{x-n-1/2}$

then $E(\varepsilon) = E^0(\varepsilon)\,h(x)$



1.6
1.4
1.2
1.0
0.8
$E(\varepsilon)/E^0(\varepsilon) = h(x)$
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$x = \frac{\varepsilon}{\hbar\omega_c}$



3.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
$dh(x)/dx$
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$x = \frac{\varepsilon}{\hbar\omega_c}$

The ground state energy for finite $H > 0$ is now $E(\varepsilon_F)$ where $\varepsilon_F = \varepsilon_F^0 + \delta\varepsilon$

$$E(\varepsilon_F) = E^0(\varepsilon_F^0 + \delta\varepsilon)\, h(x^0 + \delta x) \qquad \begin{cases} x^0 = \dfrac{\varepsilon_F^0}{\hbar\omega_c} \\ \delta x = \dfrac{\delta\varepsilon}{\hbar\omega_c} \end{cases}$$

Since $\delta\varepsilon$ is small, expand to lowest order

$$E(\varepsilon_F) \simeq \left[E^0(\varepsilon_F^0) + \frac{dE^0(\varepsilon_F^0)}{d\varepsilon}\delta\varepsilon \right]\left[h(x^0) + h'(x^0)\frac{\delta\varepsilon}{\hbar\omega_c} \right]$$

$$\simeq E^0(\varepsilon_F^0)h(x^0) + \left[E^0(\varepsilon_F^0)\frac{h'(x^0)}{\hbar\omega_c} + \frac{dE^0(\varepsilon_F^0)}{d\varepsilon}h(x^0) \right]\delta\varepsilon$$

using $\dfrac{E^0(\varepsilon)}{V} = \dfrac{3}{5}\, n \dfrac{\varepsilon^{5/2}}{(\varepsilon_F^0)^{3/2}}$ we get

$$\frac{1}{V}\frac{dE^0}{d\varepsilon} = \frac{3}{2}\, n \frac{\varepsilon^{3/2}}{(\varepsilon_F^0)^{3/2}} \quad \text{so} \quad \frac{dE^0(\varepsilon_F^0)}{d\varepsilon} = \frac{3}{2} n V$$

$$\text{also} \quad \frac{E^0}{V} = \frac{3}{5} n\, \varepsilon_F^0 \quad \text{so} \quad \frac{dE^0(\varepsilon_F^0)}{d\varepsilon} = \frac{5}{2}\frac{E^0}{\varepsilon_F^0}$$

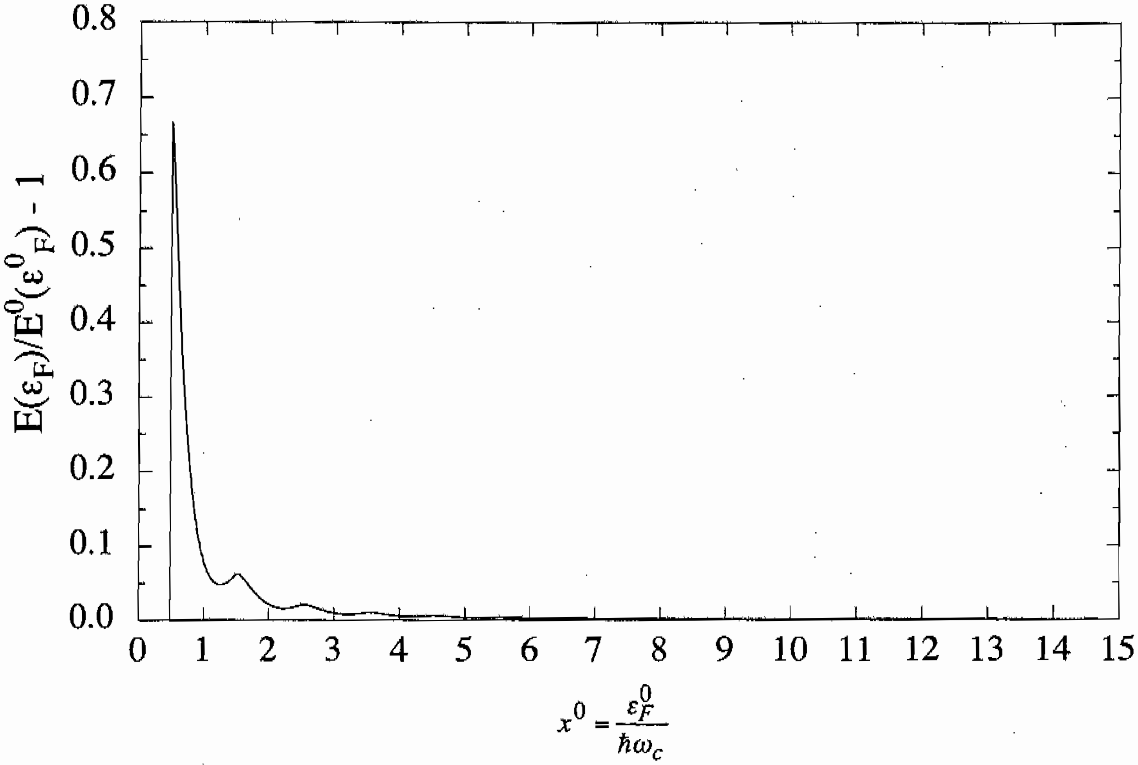
$$E(\varepsilon_F) = E^0(\varepsilon_F^0)\left[h(x^0) + \left(\frac{h'(x^0)}{\hbar\omega_c} + \frac{5}{2}\frac{h(x^0)}{\varepsilon_F^0} \right)\delta\varepsilon \right]$$

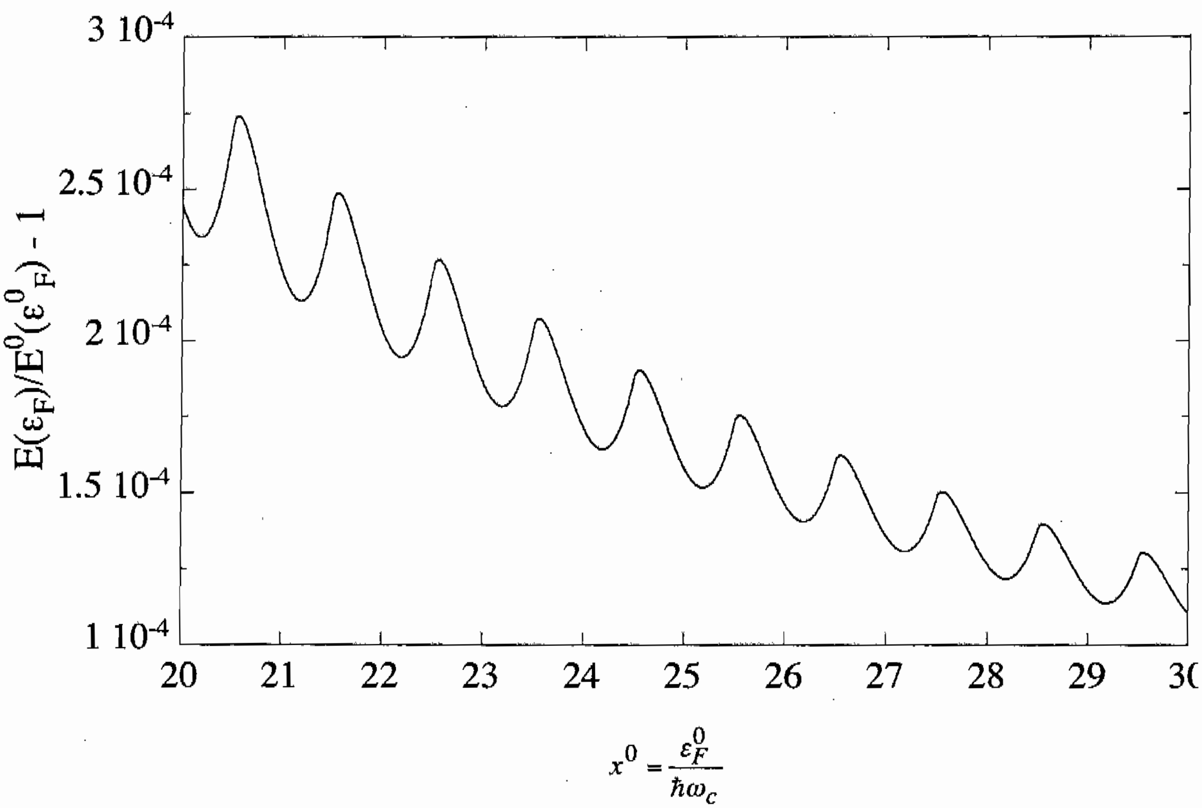
$$\boxed{\begin{array}{c} E(\varepsilon_F) = E^0(\varepsilon_F^0)\left[h(x^0) + \left(h'(x^0) + \dfrac{5}{2}\dfrac{h(x^0)}{x^0} \right)\delta x \right] \\ \text{where } x^0 = \varepsilon_F^0/\hbar\omega_c \end{array}}$$

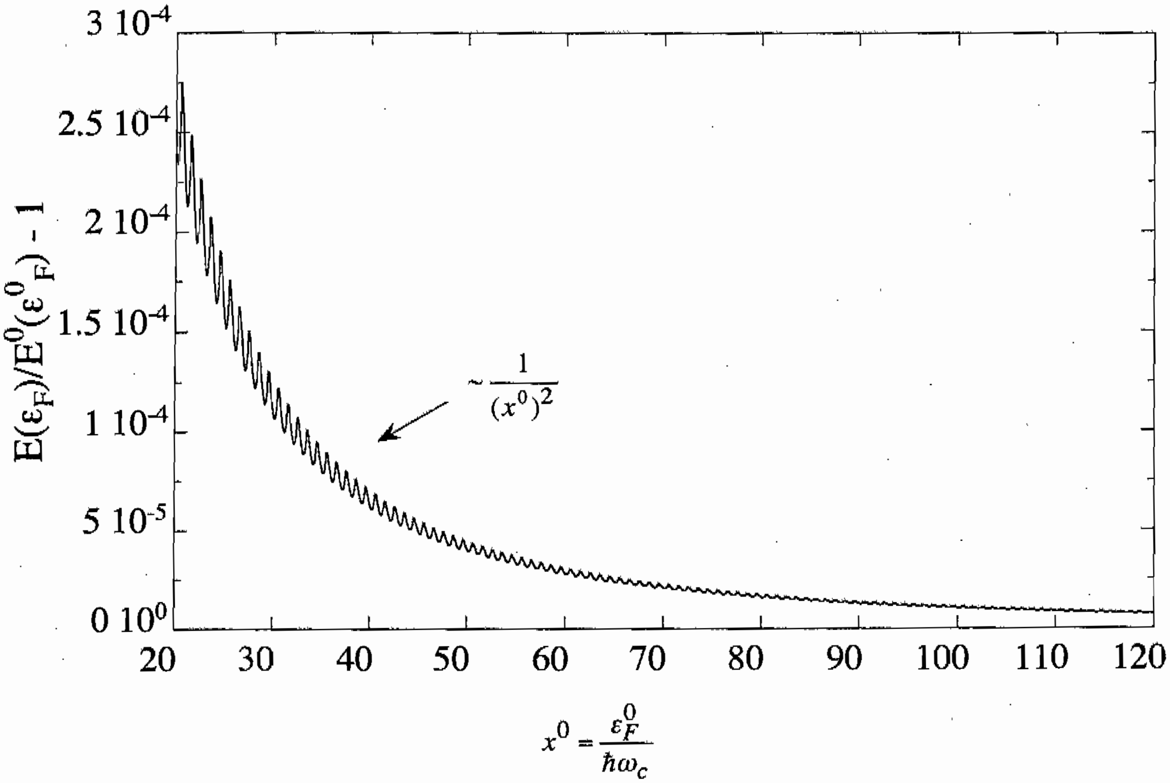
$$\text{with } \delta x = \frac{1 - f(x^0)}{f'(x^0) + \frac{3}{2}\frac{f(x^0)}{x^0}} \quad \text{as found earlier}$$

We can now plot $\dfrac{E(\varepsilon_F)}{E^0(\varepsilon_F^0)}$ as function of $x^0 = \dfrac{\varepsilon_F^0}{\hbar\omega_c}$

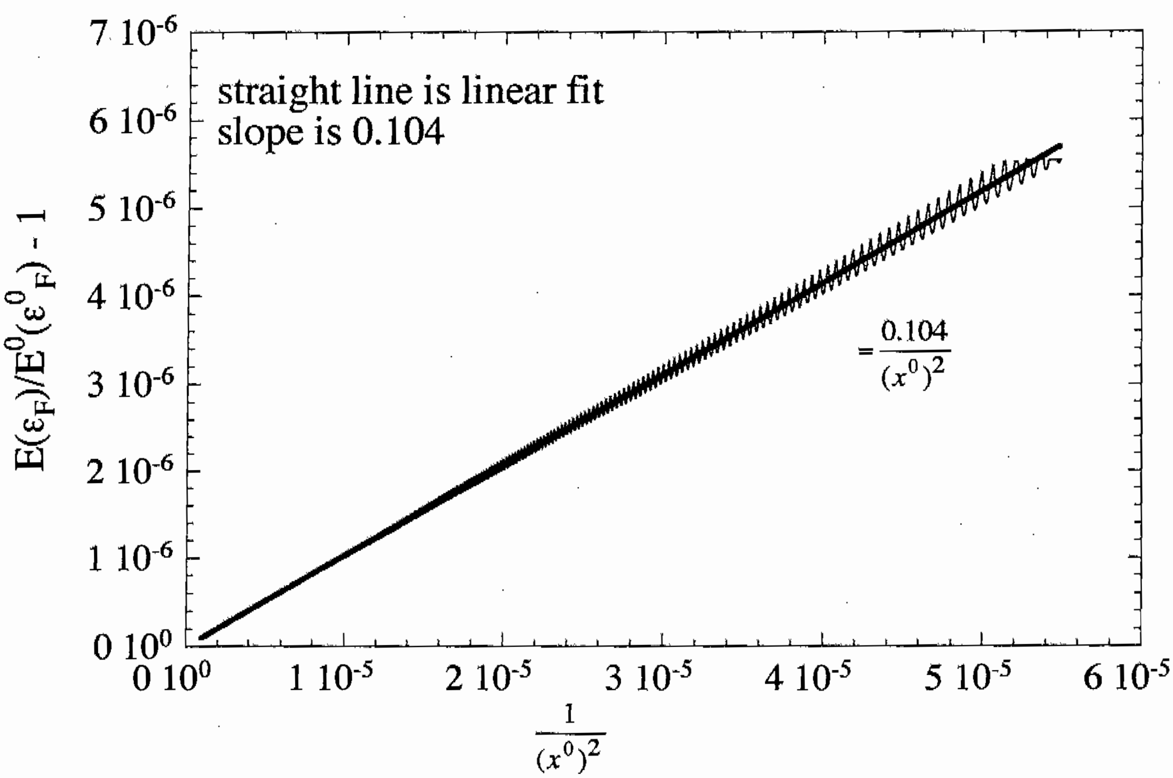
Since we expect $E(\varepsilon_F)/E^0(\varepsilon_F^0) \to 1$ as $x^0 \to \infty$, we plot $\left[\dfrac{E(\varepsilon_F)}{E^0(\varepsilon_F^0)} - 1\right]$


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$E(\varepsilon_F)/E^0(\varepsilon^0_F) - 1$
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
$x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$

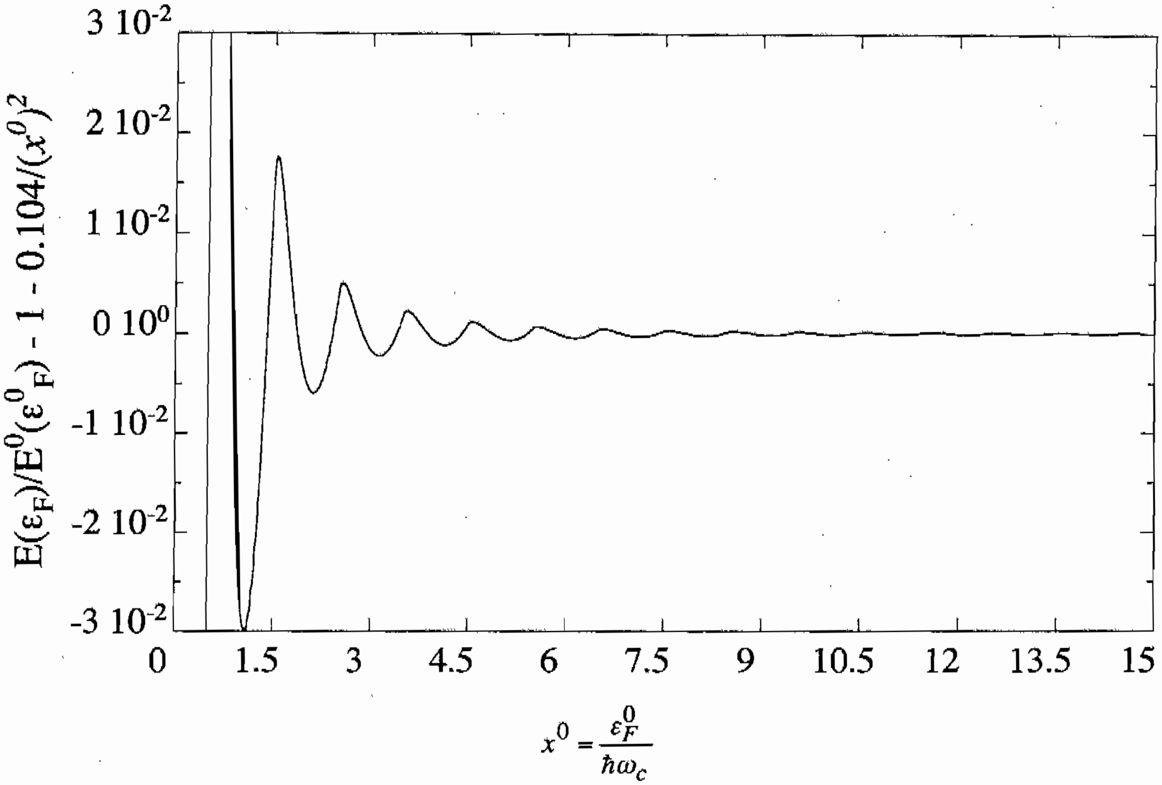

$3\ 10^{-4}$
$2.5\ 10^{-4}$
$2\ 10^{-4}$
$1.5\ 10^{-4}$
$1\ 10^{-4}$
$E(\varepsilon_F)/E^0(\varepsilon^0_F) - 1$
20 21 22 23 24 25 26 27 28 29 30
$x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$

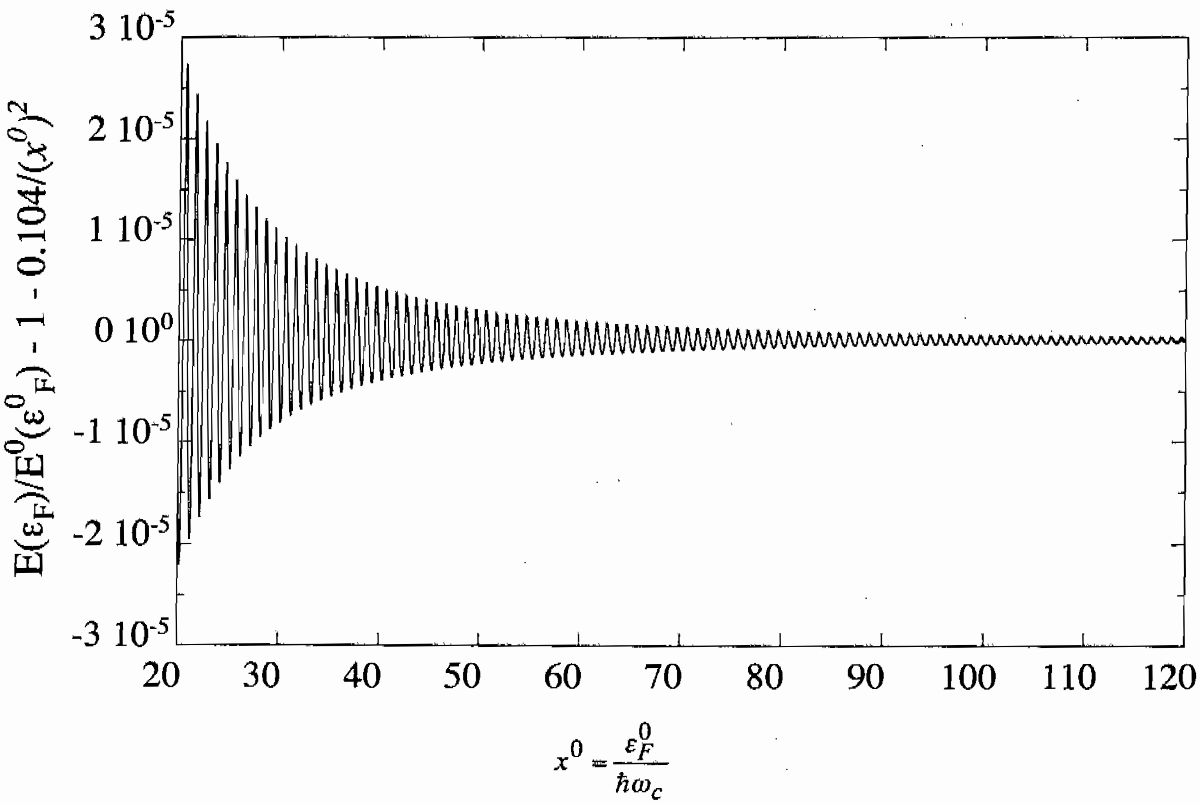


3 10-4
2.5 10-4
2 10-4
1.5 10-4
1 10-4
5 10-5
0 10^0
20
30
40
50
60
70
80
90
100
110
120
E(εF)/E0(ε0F) - 1
~ 1/(x0)^2
x0 = ε0F/ħωc



straight line is linear fit
slope is 0.104
7 10-6
6 10-6
5 10-6
4 10-6
3 10-6
2 10-6
1 10-6
0 10^0
0 10^0
1 10-5
2 10-5
3 10-5
4 10-5
5 10-5
6 10-5
E(εF)/E0(ε0F) - 1
= 0.104/(x0)^2
1/(x0)^2


3 10-2
2 10-2
1 10-2
0 10^0
-1 10-2
-2 10-2
-3 10-2
0
1.5
3
4.5
6
7.5
9
10.5
12
13.5
15
$E(\varepsilon_F)/E^0(\varepsilon^0_F) - 1 - 0.104/(x^0)^2$
$x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$


3 10-5
2 10-5
1 10-5
0 10^0
-1 10-5
-2 10-5
-3 10-5
20
30
40
50
60
70
80
90
100
110
120
$E(\varepsilon_F)/E^0(\varepsilon^0_F) - 1 - 0.104/(x^0)^2$
$x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$

From our plot of $\frac{E(\varepsilon_F)}{E^0(\varepsilon_F^0)} - 1$ we conclude that

$E(\varepsilon_F)$ has the form

$$E(\varepsilon_F) = E^0(\varepsilon_F^0)\left[1 + \frac{\alpha}{(x^0)^2} + g(x^0)\right]$$

where $\alpha = 0.104$ and $g(x)$ is a small oscillating part with period $\Delta x = 1$ and amplitude that vanishes as $x^0 \to \infty$.

At high temperature such that $k_B T \gg \hbar\omega_c$ (in practice this is anything more than a few °K) thermal fluctuations smear out the oscillations $g(x)$ and average them to zero. One is left with

$$E(\varepsilon_F) = E^0(\varepsilon_F^0) + \frac{\alpha E^0(\varepsilon_F^0)}{(x^0)^2}$$

since $x^0 = \frac{\varepsilon_F^0}{\hbar\omega_c}$ and $\omega_c = \frac{eH}{mc}$

the second term is $\propto H^2$ and so this is the term that gives the Landau diamagnetic susceptibility

$$E(\varepsilon_F) = E^0(\varepsilon_F^0) + \alpha\left(\frac{3}{5} n \varepsilon_F^0 V\right)\left(\frac{\hbar e H}{mc\,\varepsilon_F^0}\right)^2$$

where we used $E^0(\varepsilon_F^0) = \frac{3}{5} n \varepsilon_F^0 V$ V is volume

$$E(\mathcal{E}_F) = E^0(\mathcal{E}_F^0) + \frac{3}{5}\alpha \frac{m}{\mathcal{E}_F^0} V \frac{\hbar^2 e^2}{m^2 c^2} H^2$$

recall the Bohr magneton $\mu_0 = \frac{e\hbar}{2mc}$

So $E(\mathcal{E}_F) = E^0(\mathcal{E}_F^0) + \frac{12}{5}\alpha \frac{mV}{\mathcal{E}_F^0} \mu_0^2 H^2$

So $\chi_L = -\frac{1}{V} \left.\frac{\partial^2 E}{\partial H^2}\right|_{H=0} = -\frac{24}{5}\alpha \frac{m}{\mathcal{E}_F^0} \mu_0^2$

Recall that the Pauli paramagnetic susceptibility (due to intrinsic electron magnetic moment) was

$$\chi_p = \frac{3}{2} \frac{m}{\mathcal{E}_F^0} \mu_0^2$$

So

$\chi_L = -\frac{16}{5}\alpha \chi_p$ using $\alpha = .104$ we get

$\chi_L = -0.333\,\chi_p$ since $\chi_p > 0$ paramagnt. $\Rightarrow \chi_L < 0$ diamagnti

exact result $\alpha = \frac{5}{48}$

Now Landau, in his original calculation, did not have such nice computers! So he found a different method, involving a finite temperature calculation of the free energy, and using various integral approximates to discrete sums. This way he arrived at the analytical result

$$\boxed{\chi_L = -\frac{1}{3}\chi_p}$$

certainly in agreement with our numerical result.

In the preceeding calculations, we treated the paramagnetic and diamagnetic effects separately - ie, when computing Pauli paramagnetism we ignore the change in electron wavefunction due to the presence of the magnetic field H, and only considered the interaction of H with the intrinsic electron magnetic moment $\mu_0 s$. When computing Landau diamagnetism we ignored this interaction with the intrinsic moment, and considered only the effect of H on the eigenstates and hence the density of states

Of course both effects are there simultaneously, so the total magnetic susceptibility of the free electron gas is therefore

$$\chi = \chi_P + \chi_L = \chi_P - \frac{1}{3}\chi_L = \frac{2}{3}\chi_P$$

Since $\chi_P > 0$, the net effect is <u>paramagnetic</u>

For some more traditional calculations of Landau diamagnetism see:

Notes from AP Young UC Santa Cruz
http://bartok.ucsc.edu/peter/231/magnetic_field/node5.htm

Pathria - "Statistical Mechanics", pgs 206-209

Landau + Lifshitz - "Statistical Mechanics V1", pgs 172-175

The de Haas - van Alphen effect

At sufficiently low temperature and high magnetic field, so that $\hbar\omega_c > k_B T$, the oscillations due to the discrete Landau levels can be observed in measurements of magnetization $M = -\frac{1}{V}\partial E/\partial H$. These were first observed by de Haas and van Alphen in 1930 in magnetization measurements on Bi at 14.2°K. Similar oscillations are found in susceptibility $\chi = \partial M/\partial H$, conductivity (Shubnikov - de Haas effect), and many other quantities. Since we found that ε_F has such oscillations, so $g(\varepsilon_F)$ will have such oscillations, hence we can easily see why many physical quantities also oscillate.

The period of oscillations is in the inverse magnetic field $1/H$

period is $\Delta X^0 = 1 \Rightarrow \Delta\left(\frac{\varepsilon_F^0}{\hbar\omega_c}\right) = 1$ $\qquad \omega_c = \frac{eH}{mc}$

since ε_F^0 is fixed while H varies, we have oscillations that are periodic in $1/H$ with period

$$\Delta\left(\frac{1}{H}\right) = \frac{\hbar}{\varepsilon_F^0}\frac{e}{mc}$$

we can rewrite this as

$$\Delta\left(\frac{1}{H}\right) = \frac{\hbar\, 2m}{\hbar^2 k_F^2}\frac{e}{mc} = \frac{2e}{\hbar c\, k_F^2}$$

cross sectional area of the Fermi sphere is $A_F = \pi k_F^2$, so

$$\boxed{\Delta\left(\frac{1}{H}\right) = \frac{2\pi e}{\hbar c}\frac{1}{A_F}}$$

The above turns out to be more generally true. For electrons in a periodic potential (as opposed to our free electron model) the Fermi surface is not necessarily a sphere. Still the above relation holds where A_F is the ~~maximal~~ extremal cross sectional area of the Fermi surface perpendicular to the direction of the applied magnetic field. The de Haas - van Alphen effect thus became one of the methods for measuring the shape of the Fermi surface.

see Ashcroft + Mermin Chpt 14 for more details