Motion in uniform $\perp$ $\vec{E}$ and $\vec{H}$ fields
Hall effect and magnetoresistence

$$\hbar\dot{\vec{k}} = -e\left[\vec{E} + \frac{\vec{v}}{c}(\vec{k}) \times \vec{H}\right]$$

$$\Rightarrow \quad \hat{H} \times \hbar\dot{\vec{k}} = -e\,\hat{H}\times\vec{E} - \frac{eH}{c}\dot{\vec{r}}_\perp$$

$$\dot{\vec{r}}_\perp = -\frac{\hbar c}{eH}\hat{H}\times\dot{\vec{k}} + \vec{w} \qquad \vec{w} = \frac{cE}{H}(\hat{E}\times\hat{H})$$

Motion is as before, but with drift velocity $\vec{w}$ added.

To determine orbits in k space note:

$$\hbar\dot{\vec{k}} = -e\vec{E} - \frac{e}{c}\frac{1}{\hbar}\frac{\partial\mathcal{E}}{\partial\vec{k}}\times\vec{H} \qquad \text{write } \vec{E} = -(\vec{E}\times\hat{H})\times\hat{H}$$

true when $\vec{E}\perp\vec{H}$

$$= -\frac{e}{c\hbar}\left(\frac{\partial\mathcal{E}}{\partial\vec{k}} - \frac{c\hbar E}{H}\hat{E}\times\hat{H}\right)\times\vec{H}$$

$$\equiv -\frac{e}{c\hbar}\frac{\partial\bar{\mathcal{E}}}{\partial\vec{k}}\times\vec{H} \qquad \bar{\mathcal{E}} = \mathcal{E} - \hbar\vec{k}\cdot\vec{w}$$

Same as if $\vec{E}$ was absent and band structure replaced by

$$\bar{\mathcal{E}}(\vec{k}) = \mathcal{E}(\vec{k}) - \hbar\vec{k}\cdot\vec{w}$$

Orbits are intersections of surfaces of constant $\bar{\mathcal{E}}$ with planes $\perp$ to $\vec{H}$

We will assume that $-\hbar\vec{k}\cdot\vec{w}$ small enough so that if the constant $\mathcal{E}(k)$ surface is closed (open) so is the constant $\bar{\mathcal{E}}(k)$ surface. Good approx in most cases - see text for estimate of numbers.

in nearly free electron model

$\varepsilon(\vec{k}) \simeq \frac{\hbar^2}{2m} \vec{k}^2$

surface of constant energy ε is sphere of radius $\sqrt{\frac{2m\varepsilon}{\hbar^2}} = k$ in k-space

$\overline{\varepsilon}(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m} - \hbar \vec{w} \cdot \vec{k}$

surface of constant $\overline{\varepsilon}$ is given by

$$\frac{\hbar^2}{2m} \left| \vec{k} - \frac{m\vec{w}}{\hbar} \right|^2 = \overline{\varepsilon} + \frac{1}{2} m w^2$$

sphere in k-space of radius

$$k = \sqrt{\frac{2m}{\hbar^2} \left(\overline{\varepsilon} + \frac{1}{2} m w^2 \right)}$$

centered about $\vec{k}_0 = m\vec{w}/\hbar$

surface of constant $\overline{\varepsilon}$ is shifted by $\vec{w} \cdot \vec{k}$ term in direction $\vec{w}$

<u>Hall effect</u>: $\dot{\vec{r}}_\perp = -\frac{\hbar c}{eH}\,\hat{H}\times\dot{\vec{k}} + \vec{w}$, $\vec{w} = \frac{cE}{H}(\hat{E}\times\hat{H})$

current in plane $\perp$ to H is

$\vec{j} =$ ~~[illegible]~~ $- ne\langle\dot{\vec{r}}_\perp\rangle$ where $\langle\dot{\vec{r}}_\perp\rangle$ is steady state average over all occupied electron orbits, and over collisions

$\vec{j} = -ne\vec{w} + \frac{ne\hbar c}{eH}\,\hat{H}\times\langle\dot{\vec{k}}\rangle$

<u>Case (1)</u> All occupied (or unoccupied) orbits are <u>closed</u>
Then for <u>large enough H</u> so that $\omega_c\tau \gg 1$ (where τ is collision time, and $\omega_c = eH/(m^*c)$), electron makes many periods of its closed orbits between successive collisions.

We can estimate $\langle\dot{\vec{k}}\rangle$ in this large H case as follows: Averaging over electron motion between two successive collisions at $t=0$ and $t=t_0$ we get

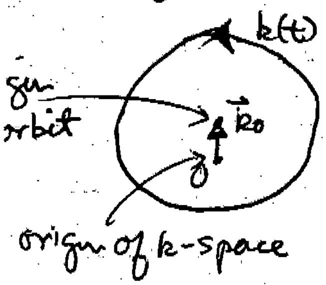
$$\langle\dot{\vec{k}}\rangle = \frac{1}{t_0}\int_0^{t_0}\dot{\vec{k}}(t)\,dt = \frac{\vec{k}(t_0)-\vec{k}(0)}{t_0}$$

where $\vec{k}(0)$ is wave vector of electron as it emerges from the first collision at $t=0$, and $\vec{k}(t_0)$ is wave vector of electron just before second collision at $t=t_0$

As in the Drude model, we may assume that electrons emerge from a collision with an equilibrium distribution determined by the local temperature + chemical potential. Since the Fermi distribution $f(\varepsilon(k)) =$ ~~[illegible]~~ $\frac{1}{1+e^{\beta(\varepsilon}}$ depends on $\vec{k}$ only via energy $\varepsilon(\vec{k})$, and $\varepsilon(\vec{k}) = \varepsilon(-\vec{k})$, we have, after averaging over the electron energy from the collision at $t=0$, $\langle \vec{k}(0) \rangle = 0$. So $\langle \dot{\vec{k}} \rangle = \vec{k}(t_0)/t_0$.

We now average over the time until the second collision, $\langle t_0 \rangle = \tau$ (this time is distributed randomly with average equal to τ). Since $\omega_c \tau \gg 1$, the electron makes many orbits between collisions, $\Rightarrow \vec{k}(t_0)$ when averaged over collision time t_0, is equally likely to lie anywhere along the closed orbit.

$\Rightarrow \langle \vec{k}(t_0) \rangle =$ (average $\vec{k}$ on orbit). If electric field $\vec{E}=0$, then (average $\vec{k}$ on orbit) $= 0$ also. But when $E \neq 0$, (average $\vec{k}$ on orbit) $\sim m^* \vec{w}/\hbar$. To see this, use effective mass approximation, $\varepsilon(k) \simeq \frac{\hbar^2 k^2}{2m^*}$. Then orbit lies on curve of constant $\bar{\varepsilon}(k) = \varepsilon(k) - \hbar \vec{k}\cdot\vec{w}$, which lies on sphere centered at $\vec{k}_0 = m^* \vec{w}/\hbar$. So (average $\vec{k}$ on orbit) $= \langle \vec{k}(t_0) \rangle = \vec{k}_0$



$$\Rightarrow \langle \dot{\vec{k}} \rangle = \frac{\langle \vec{k}(t_0) \rangle}{\tau} = \frac{\vec{k}_0}{\tau} = \frac{m^* \vec{w}}{\hbar \tau}.$$

So contribution of $\langle \dot{\vec{k}} \rangle$ term to current is

$$\frac{ne\hbar c}{eH} \hat{H} \times \frac{m^* \vec{w}}{\hbar \tau} = \frac{ne}{\omega_c \tau} \hat{H} \times \vec{w}$$

smaller than drift contribution to current $\vec{j} \simeq -ne\vec{w}$ by a factor $\frac{1}{\omega_c \tau} \ll 1$

So $\vec{j} \simeq -ne\vec{w}$ given just by drift velocity $\vec{w}$ in high field limit.

In this case $\vec{j}$ is $\parallel$ to $\vec{w}$ $\Rightarrow$ $\vec{j}$ is $\perp$ to $\vec{E}$ and $\vec{H}$
$\Rightarrow$ Lorentz force so strong that electrons move $\perp$ to E and do not acquire any energy from the E-field.

The Hall coefficient in this limit is just

$$R_{H\to\infty} = \frac{E_\perp}{jH}$$

E-field perpendicular to H (and $\perp$ to $\vec{j}$), but in this large H limit, this is just total E.

$$= \frac{E}{-newH} \qquad \text{but } \vec{w} = \frac{cE}{H}(\hat{E}\times\hat{H})$$

$$\Rightarrow R_{H\to\infty} = \frac{E}{-ne\frac{cE}{H}H} = \frac{-1}{nec}$$

Drude value

The above was for closed occupied orbits
If we had closed unoccupied orbits we would use the hole picture to get

$$R_{H\to\infty} = +\frac{1}{n_h ec} > 0$$

(n_h is density of holes, each hole has charge $+e$)

If there is more than one partially full band with only closed occupied or unoccupied orbits, then

$$\vec{j} = -n_{eff}\frac{ec}{H}(\vec{E}\times\hat{H}) \quad \text{where } n_{eff} = n - n_h$$

(n: total electron density in all partially full bands; n_h: total hole density in partial full bands)

$$R_{H\to\infty} = \frac{-1}{n_{eff}\, ec}$$

The effects of holes explains why R_∞ can have non Drude values, and even be >0.

See text for what happens when $m_{eff} = 0$. This is case for undoped semiconductor

Another way to view things is ~~not~~ in terms of conductivity tensor. keeping contribution to $\vec{j}$ from the $\langle \dot{\vec{k}} \rangle$ term gives

$$\vec{j} = -ne\vec{w} + \frac{ne}{\omega_c\tau}\hat{H}\times\vec{w} \quad , \quad \vec{w} = \frac{cE}{H}(\hat{E}\times\hat{H})$$

for $\hat{H} = \hat{z}$ direction we have

$$\vec{j} = \frac{nec}{H}\left(\hat{z}\times\vec{E} + \frac{1}{\omega_c\tau}\vec{E}\right) = \overleftrightarrow{\sigma}\cdot\vec{E}$$

with $\overleftrightarrow{\sigma} = \frac{nec}{H}\begin{pmatrix} \frac{1}{\omega_c\tau} & -1 \\ 1 & \frac{1}{\omega_c\tau} \end{pmatrix}$

or writing $\frac{\sigma_0}{\omega_c\tau} = \frac{ne^2\tau}{m^*}\,\frac{m^*c}{eH\tau} = \frac{nec}{H}$ where

σ_0 is Drude conductivity $\Rightarrow \overleftrightarrow{\sigma} = \sigma_0\begin{pmatrix} \frac{1}{(\omega_c\tau)^2} & -\frac{1}{\omega_c\tau} \\ \frac{1}{\omega_c\tau} & \frac{1}{(\omega_c\tau)^2} \end{pmatrix}$

(Compare with prob #1 on HW #1 !!)

$\Rightarrow$ resistivity tensor $\overleftrightarrow{\rho} = \overleftrightarrow{\sigma}^{-1} = \frac{1/\sigma_0}{\frac{1}{(\omega_c\tau)^4} + \frac{1}{(\omega_c\tau)^2}}\begin{pmatrix} \frac{1}{(\omega_c\tau)^2} & +\frac{1}{\omega_c\tau} \\ \frac{-1}{\omega_c\tau} & \frac{1}{(\omega_c\tau)^2} \end{pmatrix}$

~~$\Rightarrow$ Hall coefficient is $R = -\rho_{xy}/H$~~

as $\left(\frac{1}{\omega_c\tau}\right) \ll 1$

$$\overleftrightarrow{\rho} = \frac{1/\sigma_0}{1+\frac{1}{(\omega_c\tau)^2}}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} \simeq \frac{1}{\sigma_0}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix}$$

$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E} \qquad \overleftrightarrow{\sigma} = \frac{\sigma_0}{\omega_c\tau}\begin{pmatrix} \frac{1}{\omega_c\tau} & -1 \\ 1 & \frac{1}{\omega_c\tau} \end{pmatrix}$$

$$\sigma_0 = \frac{ne^2\tau}{m^*}$$

$$\omega_c\tau = \frac{eH\tau}{m^*c} \gg 1$$

Then $\vec{E} = \overleftrightarrow{\rho} \cdot \vec{j}$ where $\overleftrightarrow{\rho} \simeq \frac{1}{\sigma_0}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}$

For $\vec{j} = j\hat{x}$ then $E_y = \rho_{yx} j = -\rho_{xy} j$

Hall coef: $R = \frac{E_y}{jH} = \frac{-\rho_{xy}}{H} = \frac{-\omega_c\tau}{\sigma_0 H} = -\frac{eH\tau}{m^*c}\frac{m^*}{ne^2\tau}\frac{1}{H}$

$= -\frac{1}{nec}$ Drude value

~~For holes, $\vec{j} = n_h$~~

For electrons we used $\vec{j} = -ne\vec{w} + \frac{ne}{\omega_c\tau}\vec{H}\times\vec{w}$

For holes we use instead $\vec{j} = +n_h e\vec{w} - \frac{n_h e}{\omega_c\tau}\vec{H}\times\vec{w}$

since charge carriers have charge $+e$.

All results carry through except take $e \to -e$

$\Rightarrow R = \frac{1}{n_h ec}$

Hall coefficient is

$R \equiv \frac{-\rho_{xy}}{H}$ (see Quantum Hall effect notes)

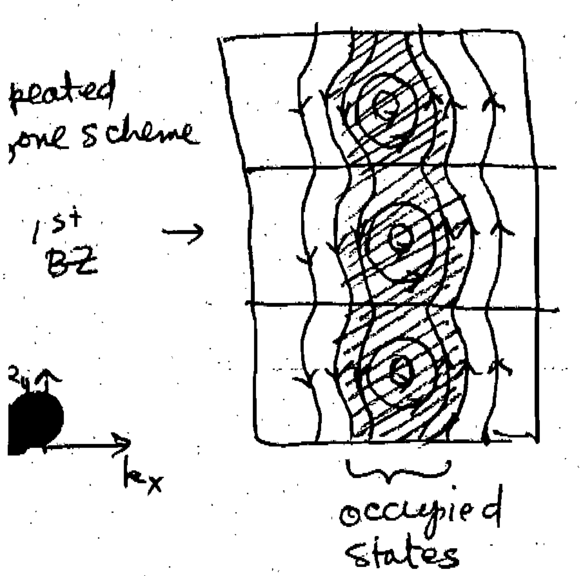
$= -\frac{\omega_c \tau}{\sigma_0 H} = -\frac{eH}{m^* c}\frac{\tau m^*}{ne^2\tau H} = -\frac{1}{nec}$ as before.

magnetoresistance

$\rho_{xx} = \rho_{xy} = \frac{1}{\sigma_0}$ saturates to finite value as $H \to 0$ just as was found in Drude model, except now m is m_{eff} if there are several partially filled bands.

Case (2) Neither all occupied states, nor all unoccupied states have closed orbits $\Rightarrow$ in either electron or hole picture there are open orbits we have to consider

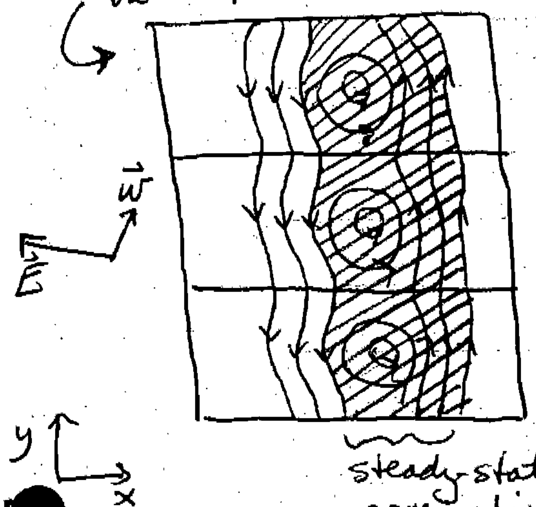
Now we will find that the $\langle \dot{\vec{k}} \rangle$ contribution to current $\vec{j}$ from these open orbits no longer vanishes in the $\omega_c \tau \to \infty$ limit, and it dominates over the drift contribution to the current $-ne\vec{w}$.



when $\vec{E}=0$, $\vec{H} = H\hat{z}$ induces motion in orbits on the constant energy surfaces. An electron moving in an open orbit in $\vec{k}$-space in the $+\hat{k}_y$ direction, gives a current in real space in the $+\hat{x}$ direction (rotate clockwise by 90° about $\hat{H}$). However when $\vec{E}=0$, each occupied open orbit going in one direction is paired with an occupied open orbit going in the opposite direction, so the net current is zero.

Note: For an open orbit traveling along $\hat{k}_y$, $k_y(t)$ is periodic in time $\Rightarrow v_y = \left\langle \frac{\partial \mathcal{E}}{\hbar \partial k_y} \right\rangle = 0$ averaged over time. But $k_x(t) \approx$ constant + oscillation $\Rightarrow v_x = \left\langle \frac{\partial \mathcal{E}}{\hbar \partial k_x} \right\rangle \neq 0 \Rightarrow$ electron moves in $\hat{x}$ direction.]

repeated zone scheme in k-space



steady state occupation

when $\vec{E} \neq 0$, in steady state, there will be an imbalance in occupation of open orbits, so that those orbits which absorb energy from the E-field have a larger population than those which lose energy to the field. ($\vec{E}$ field heats up metal!)

Open orbits in $+\hat{k}_y$ direction have real space direction $+\hat{x} \Rightarrow$ they gain energy from E field if $E_x < 0$ as energy absorbed is $\underbrace{-e\vec{E}\cdot\vec{v}\tau}_{\text{between collisions}}$.

Open orbits in $-\hat{k}_y$ directions have real space direction $-\hat{x} \Rightarrow$ they lose energy if $E_x < 0$.

$E_x < 0 \Rightarrow$ net $v_x > 0 \Rightarrow j_x < 0$

so $j_x \sim E_x$ to lowest order in E

$\vec{j} \sim \hat{x}(\vec{E}\cdot\hat{x})$

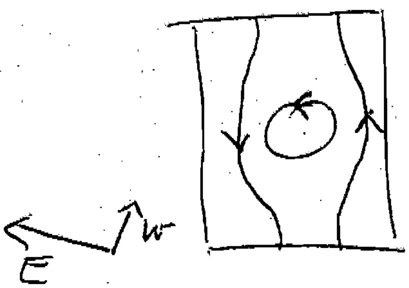
~~$\Rightarrow$ We assume therefore that the imbalance in occupation of open orbits in steady state goes like $\hat{n}\cdot\vec{E}$ gives rise to a net current. If $\hat{n}$ is the direction in real space of the open orbits, then this contribution to current $\vec{j}$ is in the $\hat{n}$ direction, and proportional to some function of $\vec{E}\cdot\hat{n}$.~~

~~$\Rightarrow \vec{j}_{\text{open orbits}} \sim \hat{n}\, g(\vec{E}\cdot\hat{n})$ expand in small $\vec{E}$,~~

Equivalently, since $\bar{\mathcal{E}} = \mathcal{E} - \hbar \vec{k}\cdot\vec{w}$ is conserved between collisions, if $\Delta\mathcal{E} = -e\vec{E}\cdot\vec{v}\,\tau$ is energy absorbed by electron from E-field then

$$\Delta\bar{\mathcal{E}} = 0 \Rightarrow \Delta\mathcal{E} = \hbar\,\vec{w}\cdot\Delta\vec{k}$$

So again we see in our example



that ~~it~~ is the ~~left~~ right hand open orbits moving along $+\hat{k}_y$ that absorb energy, ie $\vec{w}\cdot\Delta\vec{k} > 0$ for these orbits, while $\vec{w}\cdot\Delta\vec{k} < 0$ for left hand open orbits moving along $-\hat{k}_y$.

~~right hand open orbits absorb energy from field } ⇒ right hand ...~~
~~left hand open orbits lose energy to field~~

So both $\vec{w}\cdot\Delta\vec{k}$ and $-E\cdot v$ tell how much energy the electron absorbs from E-field

This imbalance in steady state occupation of open orbits is determined by the quantity $-e\vec{E}\cdot\vec{v}\tau$, the energy absorbed by electron from $\vec{E}$-field in between collisions.
If $\hat{n}$ is real space direction of open orbit, $\Rightarrow \langle\vec{v}\rangle$ is in $\hat{n}$ direction, so the current due to open orbits is in the $\hat{n}$ direction, and is some function of $(\vec{E}\cdot\hat{n})$.

$\vec{j}_{\substack{\text{open}\\\text{orbits}}} = \hat{n}\, g(\vec{E}\cdot\hat{n})$ — expand for small $\vec{E}$, using
$j = 0$ when $\vec{E} = 0$, and
$j(E) = -j(-E)$

$\vec{j}_{\substack{\text{open}\\\text{orbits}}} \sim \hat{n}\,(\hat{n}\cdot\vec{E})$ where proportionality constant is independent of magnetic field H

We can write the contribution to conductivity tensor due to open orbits as

$$\vec{j}_{\substack{\text{open}\\\text{orbits}}} = \tilde{\sigma}\cdot\vec{E} \quad \text{where } \tilde{\sigma} = \underset{\substack{\uparrow\\\text{constant indep of } H}}{\lambda}\sigma_0\,\hat{n}\,\hat{n}$$

If we choose $\hat{n}$ in $\hat{x}$ direction

$$\tilde{\sigma} = \lambda\sigma_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

If we treat the contribution to conductivity tensor from closed orbits as before, we get for total conductivity tensor

$$\overleftrightarrow{\sigma} = \frac{\sigma_0}{(\omega_c\tau)^2}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix} + \lambda\sigma_0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \sigma_0 \begin{pmatrix} \lambda + \frac{1}{(\omega_c\tau)^2} & -\frac{1}{\omega_c\tau} \\ \frac{1}{\omega_c\tau} & \frac{1}{(\omega_c\tau)^2} \end{pmatrix}$$

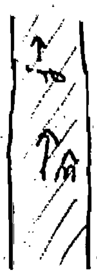
or resistivity tensor $\vec{E} = \overleftrightarrow{\rho} \cdot \vec{j}$

$$\overleftrightarrow{\rho} = \sigma^{-1} = \frac{1}{\sigma_0} \frac{1}{\left[\frac{\lambda}{(\omega_c\tau)^2} + \frac{1}{(\omega_c\tau)^2} + \frac{1}{(\omega_c\tau)^4}\right]} \begin{pmatrix} \frac{1}{(\omega_c\tau)^2} & \frac{1}{\omega_c\tau} \\ \frac{-1}{\omega_c\tau} & \lambda + \frac{1}{(\omega_c\tau)^2} \end{pmatrix}$$

$$\cong \frac{1}{\sigma_0(1+\lambda)} \begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & \lambda(\omega_c\tau)^2 + 1 \end{pmatrix}$$

Note $\rho_{xy} = \rho_{-xy}$ as before for closed orbits, and Hall coefficient is $\frac{-\omega_c\tau}{\sigma_0(1+\lambda)} = \frac{-1}{nec(1+\lambda)}$ same as before except for factor $(1+\lambda)$.

But now $\rho_{xx} \neq \rho_{yy}$. We have



expt'l wire ∥ to open orbits

ρ_{xx} – magnetoresistance for current flowing ∥ to open orbits in real space (ie $\vec{j} = j\hat{x}$)

$= \frac{1}{\sigma_0(1+\lambda)}$ ← indep of H saturates as $H \to \infty$ as in Drude model

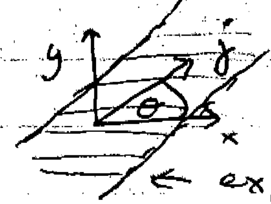
exp't'l wire ⊥ to open orbits

ρ_{yy} – magnetoresistance when current flowing ⊥ to direction of open orbits in real space (ie $\vec{j} = j\hat{y}$)

$\simeq \frac{\lambda}{\sigma_0(1+\lambda)}(\omega_c\tau)^2 \sim H^2$ does not saturate as $H \to \infty$. grows as H^2!

magnetoresistance which keeps increasing with H is signal for presence of open orbits on Fermi surface.

For a current in a general direction $\vec{j} = j\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, where θ measures angle from $\hat{x}$, the direction of the open orbits in real space



← expt'l wire at angle θ to open orbits

we have

$$\vec{E} = \overleftrightarrow{\rho} \cdot \vec{j} = \frac{j}{\sigma_0(1+\lambda)} \begin{pmatrix} \cos\theta + (\omega_c\tau)\sin\theta \\ -(\omega_c\tau)\cos\theta + (\lambda(\omega_c\tau)^2+1)\sin\theta \end{pmatrix}$$

and the longitudinal magnetoresistance is

↙ projection of $\vec{E}$ along current $\vec{j}$.

$$\rho \equiv \frac{\vec{E}\cdot\hat{j}}{|\vec{j}|}$$

$$= \frac{1}{\sigma_0(1+\lambda)}\Big[\cos^2\theta + (\omega_c\tau)\sin\theta\cos\theta - (\omega_c\tau)\cos\theta\sin\theta + [\lambda(\omega_c\tau)^2+1]\sin^2\theta\Big]$$

$$\rho = \frac{1}{\sigma_0(1+\lambda)}\Big[1 + \lambda(\omega_c\tau)^2\sin^2\theta\Big]$$

constant. Drude like part from closed orbits

$\sim H^2\sin^2\theta$ increases without bound as H increases – from open orbits