induced magnetization is in the same direction as $\vec{S}_0$, so

$$\vec{m}(\vec{r}) = +J\mu_B^2 \vec{S}_0 \, \chi(\vec{r}-\vec{R}_0)$$

For many impurities $\vec{S}_i$ at positions $\vec{R}_i$, the total induced electron magnetization is obtained from the above by superposition

$$\vec{m}(\vec{r}) = +J\mu_B^2 \sum_i \vec{S}_i \, \chi(\vec{r}-\vec{R}_i)$$

The interaction Hamiltonian is then

$$\delta H = J \sum_j \vec{S}_j \cdot \vec{m}(\vec{R}_j)$$

$$\boxed{\delta H = +J^2\mu_B^2 \sum_{i,j} \vec{S}_j \cdot \vec{S}_i \, \chi(\vec{R}_j - \vec{R}_i)}$$

Above result shows how the magnetization of the conduction electrons mediates an interaction between the two magnetic impurities $\vec{S}_i$ and $\vec{S}_j$. If $\chi(\vec{R}_j-\vec{R}_i) < 0$ then the interaction is ferromagnetic. If $\chi(\vec{R}_j-\vec{R}_i) > 0$ then the interaction is antiferromagnetic.

Now $\chi(\vec{r}) = \int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, \chi(\vec{q})$

with $\chi(\vec{q}) = 2\int \frac{d^3k}{(2\pi)^3} \left[ \frac{f_{k+q} - f_k}{\varepsilon_{k+q} - \varepsilon_k} \right]$

$$= -g(\varepsilon_F)\left[ 1 + \frac{1-x^2}{2x} \ln\left|\frac{1+x}{1-x}\right| \right]$$

where $x = q/2k_F$

As discussed in connection with the Lindhard dielectric function, $\chi(\vec{q})$ has a singularity at $x=0$ or $|\vec{q}| = 2k_F$. This results in $\chi(\vec{r})$ having a piece that goes as

$$\chi(\vec{r}) \sim \frac{1}{r^3}\cos(2k_F r)$$

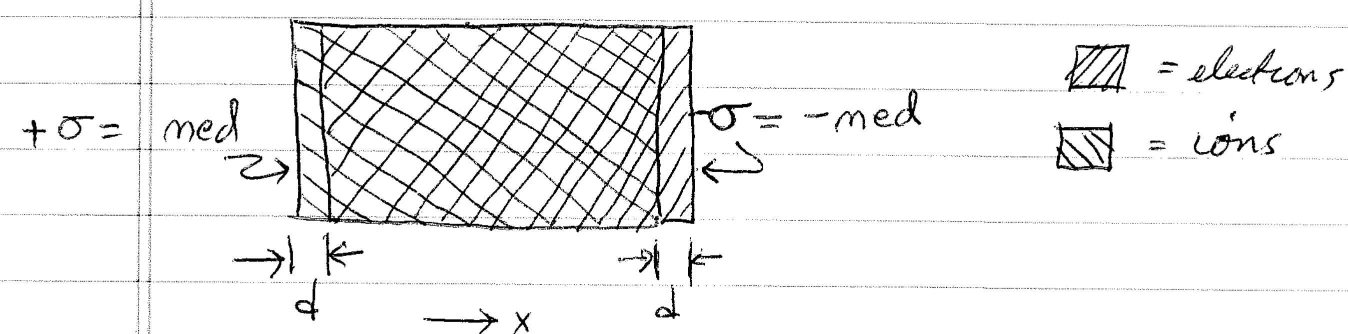
which oscillates in sign depending on the value of the distance $r$. Since the magnetic impurities $\vec{S}_i$ are randomly positioned in the metal, with an average spacing several times the atomic lattice constant, then $k_F|\vec{R}_i - \vec{R}_j|$ in general is large and hence $\chi(\vec{R}_i - \vec{R}_j)$ will be randomly positive or negative, according to the particular random separation between the spins. Thus the interaction between spins $\vec{S}_i$ and $\vec{S}_j$ is randomly ferro or anti-ferro magnetic. This is the the model interaction for a "spinglass" where the spins freeze into random orientations as $T$ decreases

## Plasmon

Although we argued by screening that e-e interactions are less important than one might naively expect, nevertheless the Coulomb interaction between electrons does give rise to physically interesting effects.

One such effect is the plasmon - which is a longitudinal charge density oscillation.

Simple explanation: consider the gas of electrons as a rigid charged body of mass $mN = mnV$ where $N$ is the total number of electrons. If we displace the electrons a distance $d$ with respect to the ions, we will create a surface charge on the surfaces of the system as shown below.




surface charge $\sigma$ creates electric field inside

$$\vec{E} = 4\pi\sigma\hat{x} = 4\pi med\,\hat{x}$$

Newton's equation of motion for the electrons is then

$$mN\ddot{d} = -eNE = -4\pi Me^2 dN$$

$$\ddot{d} = -\frac{4\pi Me^2}{m} d$$

→ harmonic oscillation at frequency $\boxed{\omega_p = \sqrt{\frac{4\pi Me^2}{m}}}$

the plasma frequency!

⇒ oscillation in charge and $\vec{E}$ with freq $\omega_p$.

<u>Another way to get plasma oscillations</u> from Maxwell's equations

When we considered EM wave propagation in a metal early in the course, we limited discussion to <u>transverse</u> modes where $\vec{k}\cdot\vec{E}=0$. The plasma oscillation is a <u>longitudinal</u> mode $\vec{k}\cdot\vec{E}\neq 0$.

charge conservation: $\vec{\nabla}\cdot\vec{j} = -\frac{\partial \rho}{\partial t}$

for harmonic oscillation: $\vec{j} = \vec{j}_0 e^{-i\omega t} e^{i\vec{k}\cdot\vec{r}}$

freq $\omega$, wavevector $\vec{k}$ $\quad \rho = \rho_0 e^{-i\omega t} e^{i\vec{k}\cdot\vec{r}}$

$$\Rightarrow i\vec{k}\cdot\vec{j}_0 = i\omega\rho_0$$

But we also had $\vec{j}_0 = \sigma(\omega)\vec{E}_0$ $\quad$ $\sigma$ is ac conductivity

$$\Rightarrow i\vec{k}\cdot\sigma\vec{E}_0 = i\omega\rho_0$$

From Gauss's Law $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$

$$\Rightarrow \quad i\vec{k} \cdot \vec{E}_0 = 4\pi\rho$$

Combine above with charge conservation to get

$$\frac{\sigma}{\omega} \vec{k} \cdot \vec{E}_0 = \frac{i \vec{k} \cdot \vec{E}_0}{4\pi}$$

~~Thus~~ If there is to be a solution, then either

$$\vec{k} \cdot \vec{E}_0 = 0 \quad \Rightarrow \text{transverse mode}$$

or

$$\frac{4\pi\sigma}{i\omega} = 1$$

$$\Rightarrow \quad \boxed{1 + \frac{4\pi i \sigma}{\omega} = 0}$$

We saw the above quantity earlier in our discussion of transverse wave propagation in metals. Then we had for the dispersion relation for the transverse EM waves:

$$k^2 = \frac{\omega^2}{c^2}\left[1 + \frac{4\pi i \sigma}{\omega}\right]$$

In analogy with dielectrics, one sometimes ~~writes~~ defines

$$\varepsilon(\omega) = 1 + \frac{4\pi i \sigma(\omega)}{\omega} \quad \text{for a metal}$$

$\uparrow$

complex ~~dielectr~~ frequency dependent dielectric function

Longitudinal
~~Plasma~~ oscillations occur when

$$\varepsilon(\omega) = 1 + \frac{4\pi i \sigma(\omega)}{\omega} = 0$$

From our discussion of the Drude model we had

$$\sigma(\omega) = \frac{\sigma_{dc}}{1 - i\omega\tau} \qquad \sigma_{dc} = \frac{ne^2\tau}{m}$$

For high frequencies $\underline{\underline{\omega\tau \gg 1}}$, $\sigma(\omega) \simeq \frac{\sigma_0}{-i\omega\tau}$

$$= \frac{ne^2}{-i\omega m}$$

and so

$$\varepsilon(\omega) = 1 - \frac{4\pi n e^2}{m\omega^2}$$

$$= 1 - \left(\frac{\omega_p}{\omega}\right)^2 \quad \text{with} \quad \omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

So the condition $\varepsilon(\omega) = 0$ for longitudinal modes of oscillation

$$\Rightarrow \boxed{\omega = \omega_p} \quad \text{for } \cancel{\text{all}} \text{ any wavevectors } \vec{k}$$

Such longitudinal modes are called "plasma" oscillations since ~~they are accompanied~~ the longitudinal oscillations of the electric field ($\vec{k} \cdot \vec{E}_0 \neq 0$) are (by Gauss' law) accompanied of oscillations in electron charge density.

Note, the above Maxwell eqn arguement gives a plasma oscillation at $\omega = \omega_p$ for any longitudinal wave vector $\vec{k}$. In reality, the ~~plasma freque~~ frequency of plasma oscillations does depend on $\vec{k}$.

In our derivation of $\sigma(\omega)$ we assumed that the wavelength $\lambda$ of the EM oscillations was macroscopically large, ie $\gg$ atomic lengths. This lead to a $\sigma(\omega)$ independent of wavevector $\vec{k}$. (ie we ignored spatial dependence of $\vec{E}$ on equation of motion of electron). When one does a better job, one finds that $\varepsilon = 1 + 4\pi i \sigma(\omega)/\omega$ should really have a dependence on $\vec{k}$ as well, that is important when $k$ is of the order $1/a_0$, ie $\lambda \sim a_0$ atomic length scale. (Recall the $k$-dependence of the Thomas-Fermi dielectric function for the $\omega = 0$ case). If one includes this $k$ dependence of $\varepsilon(\vec{k}, \omega)$, then the condition $\varepsilon(\vec{k}, \omega) = 0$ gives a dispersion relation for plasma oscillations:

$$\omega_p(\vec{k}) \simeq \omega_p \left[ 1 + \frac{3}{10} \frac{v_F^2 k^2}{\omega_p^2} \right]$$

where $\omega_p = \sqrt{4\pi n e^2/m}$ as before
and $v_F$ is the Fermi velocity

Note $$\frac{v_F^2 k^2}{\omega_p^2} = 4 \left(\frac{\varepsilon_F}{\hbar\omega_p}\right)^2 \left(\frac{k}{k_F}\right)^2$$

For Typical metals, $\varepsilon_F \sim 2-10$ eV

$\hbar\omega_p \sim 10-20$ eV

$\Rightarrow$ correction to $\omega_p$ at finite $k$ is usually quite small for $k < k_F$.

As with other harmonic oscillations, the longitudinal plasma oscillations of electrons in a metal, get quantized in a more complete quantum mechanical treatment of the EM fields. When so quantized, the plasma oscillations are refered to as "plasmons". ~~and have energy~~ The energy associated with the $n^{th}$ level of excitation of the oscillations with wavevector $\vec{k}$, ie the energy of $n$ plasmons of wave vector $\vec{k}$, is just $(n+1/2)\hbar\omega_p(\vec{k})$.

Because $\hbar\omega_p \sim 10-20$ eV $\gg k_BT$, plasmons are not in general thermally excited. However the zero point energy of the plasmon modes, ie the $\frac{1}{2}\hbar\omega_p(\vec{k})$, does contribute to the ~~ground~~ ~~stat~~ total ground state energy of the electron gas.

When one shoots a high energy electron into a metal surface, one can see energy loses corresponding to the excitation of integer numbers of plasmons with energies $n\hbar\omega_p$.

## Another moral from the story of the plasmon:

We start with electrons which are fermions.
A bare electron has energy $\varepsilon(k) = \frac{\hbar^2 k^2}{2m}$.

When we include effects of the Coulomb interactions among the electrons in a gas of electrons, we get not only fermionic degrees of freedom with dispersion relation $\varepsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m}$, but now we also get bosonic degrees of freedom, ie the plasmons with dispersion relation

$$\hbar\omega_p(\vec{k}) \simeq \hbar\omega_p\left(1 + \frac{3}{10}\frac{v_F^2 k^2}{\omega_p^2}\right)$$

(goes to constant $\omega_p$ as $k \to 0$.
weak dependence on $k$ for small $k < k_F$.

Moral: The presence of strong interactions among the "bare" (ie isolated) degrees of freedom can lead to elementary excitations (ie new degrees of freedom) of the system that bear no resemblance at all to the bare degrees of freedom – ie they can have a completely different dispersion relation $\varepsilon(\vec{k})$ and can even have different symmetry, ie bosonic instead of fermionic. This is a general rule to remember in all fields of physics! (Another condensed matter example is phonons: bare ions have $\varepsilon(k) = \frac{\hbar^2 k^2}{2M}$. But the interacting ions lead to quantized elastic vibrations (phonons) with $\hbar\omega(\vec{k}) \sim c\hbar k$ – sound modes).