2.2. THERMODYNAMICS OF PERFECT GASES

Symbol	Explanation		cgs unit
P	gas pressure		dynes cm^{-2}
ρ	gas density		g cm^{-3}
n	gas concentration (or number density)		particles cm^{-3}
μ	mean molecular mass		g
k	Boltzmann's constant	1.38×10^{-16}	erg K^{-1}
T	gas temperature		K
v	volume per unit mass $(1/\rho)$		$\mathrm{cm}^3\ \mathrm{g}^{-1}$
c_P	specific heat per unit mass at constant pressure		erg $\mathrm{K}^{-1}\ \mathrm{g}^{-1}$
c_V	specific heat per unit mass at constant volume		erg $\mathrm{K}^{-1}\ \mathrm{g}^{-1}$
γ	ratio of specific heats, c_P/c_V		dimensionless
e	internal energy per unit mass		erg g^{-1}
δQ	heat input per unit mass		erg g^{-1}
s	entropy per unit mass		erg $\mathrm{K}^{-1}\ \mathrm{g}^{-1}$
u	internal energy per unit volume		erg cm^{-3}

Extensive quantities (i.e., those like energy, where twice as much gas implies twice as much of the quantity) will be expressed as amounts per unit mass, e.g., $e(T)$ = internal energy per unit mass of gas, a measure of the kinetic energy of the molecules and atoms.

2.2.1. Ideal Gas Law

For independent, noninteracting, particles of infinitesimal size, whose pressure comes entirely from random motions and impacts with the walls,

$$P = nkT \ ,$$

or, since $n = \rho/\mu$,

$$\boxed{P = \frac{\rho k T}{\mu} \ ,} \tag{2.2.1}$$

or, since $v = 1/\rho$,

$$Pv = \frac{kT}{\mu} \quad .$$

All three forms of the law are useful: the second probably the most so.

2.2.2. Zeroth Law of Thermodynamics

"Hot things lose heat to cool things until their temperatures are equal,"

i.e., a quantity called temperature (T) tells you how heat will flow.

2.2.3. First Law of Thermodynamics

"Heat is work and work is heat,"

i.e., heat input to gas $\Longrightarrow$ (increase in random motion of particles)
+ (work done changing volume of gas)

i.e.,

$$\boxed{\delta Q = de + Pdv \quad .} \tag{2.2.2}$$

$\delta Q =$ small amount of energy input (not a function of the *state* of the gas, therefore written as δQ, not dQ)

$de =$ change in internal energy of gas (per unit mass)

$Pdv =$ work done (per unit mass of gas) when pressure P forces volume to increase by dv

Now $e = e(T)$ only: the kinetic energy of molecules in the gas, and the rotational and vibrational energy of atoms in the molecules, are functions only of temperature. Therefore,

$$\delta Q = \left(\frac{de}{dT}\right) dT + Pdv \quad .$$

But if we put energy δQ into the gas, holding its volume constant, the temperature rise, dT, is given by

$$\delta Q = c_V \, dT \quad ,$$

where c_V is the specific heat per unit mass at constant volume. Thus,

$$\boxed{\frac{de}{dT} = c_V \quad .} \tag{2.2.3}$$

Now, this means that we can write the first law in the form

$$\delta Q = c_V \, dT + P dv \quad ,$$

but for an ideal gas

$$Pv = \frac{kT}{\mu} \quad ,$$

so, differentiating,

$$P dv = \frac{k}{\mu} dT - v dP \quad ,$$

and therefore, in the first law,

$$\delta Q = \left(c_V + \frac{k}{\mu} \right) dT - v dP \quad .$$

But if we put energy δQ into the gas, holding its pressure constant, the temperature rise is given by

$$\delta Q = c_P \, dT \quad ,$$

where c_P is the specific heat per unit mass at constant pressure. Therefore, the specific heats are related by

$$\boxed{c_P = c_V + \frac{k}{\mu} \quad .} \tag{2.2.4}$$

Define

$$\gamma = \frac{c_P}{c_V} = \text{ratio of specific heats} \quad .$$

Then

$$\boxed{\begin{aligned} c_P &= \frac{k}{\mu} \cdot \frac{\gamma}{\gamma - 1} \quad . \\ c_V &= \frac{k}{\mu} \cdot \frac{1}{\gamma - 1} \quad . \end{aligned}} \tag{2.2.5}$$

Note the units of c_P and c_V: erg g^{-1} K^{-1}, or J kg^{-1} K^{-1}.

From equation (2.2.2),

$$de = c_V\, dT \quad ,$$

so that if c_V is a constant,

$$e = c_V T + \text{constant} \quad .$$

Define the zero of internal energy to occur at the zero of *absolute temperature.* Then

$$\boxed{e = c_V T = \frac{kT}{\mu(\gamma - 1)} \quad .} \tag{2.2.6}$$

We'll sometimes also use u, the internal energy per unit volume:

$$\boxed{u = \rho e = \frac{\rho k T}{\mu(\gamma - 1)} = \frac{P}{\gamma - 1} \quad .} \tag{2.2.7}$$

(Note that this says that *the pressure of a gas arises directly from its internal energy density.*)

2.2.4. Second Law of Thermodynamics

"Heat cannot, of itself, pass from one body to a hotter body,"

i.e., some processes that are allowed by conservation of energy (the First Law) nevertheless do not occur. This leads to the idea of entropy: define the *entropy per unit mass*, s, by

$$ds = \frac{\delta Q}{T} \quad ,$$

then, from the first law [equations (2.2.1) and (2.2.2)]

$$\begin{aligned} ds &= c_V \frac{dT}{T} + \frac{P}{T}\, dv \\ &= c_V \frac{dT}{T} + \frac{k}{\mu} \cdot \frac{dv}{v} \quad . \end{aligned}$$

Thus,

$$s = \text{constant} + c_V \ln T + \frac{k}{\mu} \ln v \quad .$$

Use $c_V = \frac{k}{\mu} \cdot \frac{1}{\gamma - 1}$ [equation (2.2.4)];

$$\boxed{\begin{aligned} s &= (\text{constant})_1 + c_V \ln(Tv^{\gamma-1}) \\ &= (\text{constant})_2 + c_V \ln(Pv^{\gamma}) \\ &= (\text{constant})_3 + c_V \ln(P\rho^{-\gamma}) \ . \end{aligned}} \tag{2.2.8}$$

The last of these will be the most generally useful to us. Engineers sometimes say that s reflects the number of lost opportunities for using the gas to do work. You may also like to remember that s measures the *degree of disorder* of the gas – the number of ways the molecules can share energy.

2.2.5. Third Law of Thermodynamics

"Everything is a perfect crystal at $T = 0$,"

$T = 0 \Longrightarrow$ perfect order $\Longrightarrow$ zero entropy

This gives the value of the constant in the $s(P, \rho)$ relations [equation (2.2.8)], but the third law will not be of much importance in our use of thermodynamics.

2.2.6. Stellar Structure Virial Theorem

Assuming a star is a perfect gas in a self-gravitating sphere, we can use these thermodynamic results and the pressure equation to deduce a virial theorem relating to stellar structure. Start from the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)\,\rho(r)}{r^2}$$

and define $V(r)$ = volume occupied by gas inside radius $r = \frac{4}{3}\pi r^3$, so that dV = volume of dr-shell, containing mass $\rho(r)\,4\pi r^2 dr$.

Multiply the pressure equation by $V(r)\,dr$:

$$V(r)\frac{dP}{dr}\,dr = -\frac{GM(r)\,\rho(r)}{r^2}\,V(r)\,dr \ ,$$

i.e.,

$$V(r)\,dP = -GM(r)\,\rho(r)\,\frac{4}{3}\pi r^3 dr\,\frac{1}{r^2} = -\frac{1}{3}GM(r)\,dM(r)\,\frac{1}{r} \ .$$