Convection

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1 Convection Instability

(AST 462, Eric Blackman's lecture note)
Consider a perfect gas in hydro-static equilibrium in uniform gravity. If z axis is chosen such that gravity is in negative z direction, then $g(z)$ and $\rho(z)$ decreases with z. Consider the vertical displacement of the blob as shown Where initially p and ρ have the same density as surroundings, external density and pressure at new position are p' and ρ'. Pressure balences inside and outside is maintained swiftly by acoustic waves, but heat imbalence takes longer when mediated by conduction. We can consider the blob to be displaced adiabatically, then let ρ^* be its new density. if $\rho^* < \rho'$, the blob will be bouyant and continue upward, implying instability. if $\rho^* > \rho'$, then the blob will tend to return, making the system stable. So we need to determine ρ^*/ρ. For adiabatic flow,

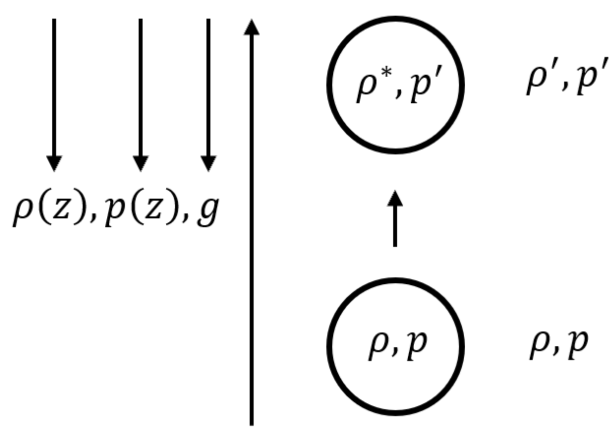
$$\rho^* = \rho\Big(\frac{p'}{p}\Big)^{1/\gamma} \tag{1}$$

If $\frac{dp}{dz}$ is the pressure gradient, then

$$p' = p + \frac{dp}{dz}\Delta z \tag{2}$$

and using

$$\rho^* = \rho\Big(\frac{p'}{p}\Big)^{1/\gamma} = \rho\Big(\frac{p + \frac{dp}{dz}\Delta z}{p}\Big)^{1/\gamma} \approx \rho + \frac{\rho}{\gamma p}\frac{dp}{dz}\Delta z \tag{3}$$



The last step is obtained by expanding to lowest order in Δz
But for ambient medium,

$$\rho' = \rho + \frac{d\rho}{dz}\Delta z \tag{4}$$

Then, using $\rho = \frac{p}{RT}$

$$\rho' = \rho + \frac{d\rho}{dp}\frac{dp}{dz}\Delta z + \frac{d\rho}{dT}\frac{dT}{dz}\Delta z = \rho + \frac{\rho}{p}\frac{dp}{dz}\Delta z - \frac{\rho}{T}\frac{dT}{dz}\Delta z \tag{5}$$

Using equation 3 and 5, we get

$$\rho^* - \rho' = \frac{\rho}{\gamma p}\frac{dp}{dz}\Delta z - \frac{\rho}{p}\frac{dp}{dz}\Delta z + \frac{\rho}{T}\frac{dT}{dz}\Delta z = \rho\Big[\Big(\frac{1}{\gamma} - 1\Big)\frac{1}{p}\frac{dp}{dz} + \frac{1}{T}\frac{dT}{dz}\Big]\Delta z \tag{6}$$

For stable atmosphere, we require that $\rho^* > \rho'$, thus

$$\Big(\frac{1}{\gamma} - 1\Big)\frac{1}{p}\frac{dp}{dz} + \frac{1}{T}\frac{dT}{dz} > 0 \quad \Rightarrow \quad \frac{dT}{dz} > \Big(1 - \frac{1}{\gamma}\Big)\frac{T}{p}\frac{dp}{dz} \quad \Rightarrow \quad \Big|\frac{dT}{dz}\Big| > \Big(1 - \frac{1}{\gamma}\Big)\frac{T}{p}\Big|\frac{dp}{dz}\Big| \tag{7}$$

The last step because $\frac{dT}{dz} < 0$ and $\frac{dp}{dz} < 0$
This is called the "Schwarzschild stability condition"
Force per unit volume inside the displaced blob is approximatly $(\rho^* - \rho)(-g)$, the equation of motion is therefore

$$\rho^*\frac{d^2\Delta z}{dt^2} = -(\rho^* - \rho')g = -g\rho\Big[\Big(\frac{1}{\gamma} - 1\Big)\frac{1}{p}\frac{dp}{dz} + \frac{1}{T}\frac{dT}{dz}\Big]\Delta z \tag{8}$$

Or equivalently

$$\frac{d^2\Delta z}{dt^2} + N'^2\Delta z = 0 \tag{9}$$

Where

$$N' = \sqrt{\frac{g}{T}\frac{\rho}{\rho^*}\frac{dT}{dz} - \Big(1 - \frac{1}{\gamma}\Big)\frac{g}{p}\frac{\rho}{\rho^*}\frac{dp}{dz}} \tag{10}$$

When $\rho \approx \rho^*$ for lowest order of Δz, we get

$$N \equiv \sqrt{\frac{g}{T}\frac{dT}{dz} - \Big(1 - \frac{1}{\gamma}\Big)\frac{g}{p}\frac{dp}{dz}} \tag{11}$$

called the "Brunt-Vaisula frequency". For stable equilibrium, the blub will oscillate.
In reality such motion give rise to internal gravity waves by disturbing the surrounding medium. We ignore internal gravity waves by ignoring the effect of the blub motion on external medium. Full treatments account for these waves when full perturbative treatment is developed (?)

2 Equations of Stellar Evolution

(Ohlmann Thesis sec. 2.4.1)
In a stationary spherical configuration, the mass profile of the star is given by

$$\frac{\partial m(r,t)}{\partial r} = 4\pi r^2\rho(r,t) \tag{12}$$

This is a consequence of the geometry. Boundary condition: $m(0) = 0$.
For hydrostatic equilibrium, we need the gravitational acceleration $g = \frac{GM(r)}{r^2}$ for spherical mass distribution. In hydrostatic equilibrium, this self-gravity balences the force from pressure gradient

$$\frac{\partial p(r,t)}{\partial r} = -\frac{Gm(r,t)}{r^2}\rho(r,t) \tag{13}$$

Boundary condition: $p(R) = 0$, where $r = R$ is the surface of the star. This BC can be replaced by a more sophisticated treatment.
To solve this system of equations, an additional relation has to be supplied: the equation of state. The internal structure may not be determined by only considering the mechanical equilibrium, but also the thermal equilibrium that depends on energy sources or sinks and energy transport.
Let $l(r,t)$ be the net energy flux through a sphere of radius r, then energy conservation demands

$$\frac{\partial l(r,t)}{\partial r} = 4\pi r^2 \rho(r,t)\big(\epsilon_n - \epsilon_\nu + \epsilon_g\big) \tag{14}$$

where ϵ_n = nuclear reactions, ϵ_ν = neutrino losses and ϵ_g = mechanical work due to contraction or expansion.
B.C.: At the center of the star $l(0) = 0$; at the surface: $l(R) = L$, the luminosity of the star.
The distribution of the temperature depends on the way energy is transported in the star - either by radiation or convection - and may be written as

$$\frac{\partial T(r,t)}{\partial r} = -\frac{Gm(r,t)}{r^2}\frac{T(r,p)}{p(r,t)}\frac{d\ln T}{d\ln p}\rho(r,t) \tag{15}$$

Is Ohlmann missing by a factor of ρ here? because

$$\frac{\partial T(r,t)}{\partial r} = -\frac{\partial T}{\partial p}\frac{\partial p}{\partial r} = -\frac{Gm}{r^2}\rho\frac{\partial T}{\partial p} = -\frac{Gm\rho}{r^2}\frac{\partial T}{T}\frac{p}{\partial p}\frac{T}{p} = -\frac{Gm\rho}{r^2}\frac{T}{p}\frac{\partial \ln T}{\partial \ln p} \tag{16}$$

where $m = m(r,t)$; $T = T(r,t)$, $p = p(r,t)$
Where $\boldsymbol{\nabla} \equiv \frac{d\ln T}{d\ln p}$ denotes the temperature gradient that depends on energy transportation mechanism. If energy transport by radiation dominates, a diffusion approximation leads to

$$\boldsymbol{\nabla}_{\text{rad}} = \frac{3}{16\pi acG}\frac{\kappa lp}{mT^4} \tag{17}$$

where a = radiation constant, c = speed of light, κ = opacity.
If convection is the dominant energy transport mechanism, $\boldsymbol{\nabla}$ has to be supplied by a theory of convection, which is usually the mixing-length theory. In the deep interior it is usually near the adiabatic value $\boldsymbol{\nabla}_{\text{ad}}$.
The chemical composition is given by the mass fractions $X_i(r,t)$ for species i, These change due to nuclear reactions and due to mixing in convective regions.

$$\frac{dX_i(r,t)}{dt} = \frac{m_i}{\rho}\Big(\sum_j r_{ji} - \sum_k r_{ik}\Big) \tag{18}$$

Where the derivative is taken in the Lagrangian frame to take into account mixing process. Here m_i denotes the mass of nucleus i, and r_{ij} is the reaction rate from nucleus i to nucleus j. Moveover, the composition is assumed to be homogeneous in convective regions.
(Ohlmann Thesis sec. 2.4.3)
Most important hydrodynamical instabilities is convective instability caused by buoyancy. Suppose a fluid element embedded in a stellar profile rises adiabatically. If at new radius the buoyancy force is larger compared to the surroundings, the fluid element will rise further; thus ht stratification is unstable. The buoyancy force per unit volume is given by

$$\mathbf{F} = \mathbf{g}\cdot((\Delta\rho)_e - (\Delta\rho)_s)\Delta r\hat{\mathbf{r}} \tag{19}$$

g is gravitational acceleration, the indices denote the quantities in the element (e) and surroundings (s), Δr denotes the radial shift. Since the gravity is antiparallel to the density gradient, the difference of the density gradients has to be larger than zero for the right-hand side to be negative; this means that the buoyancy

force points downwards, the fluid element is pulled back, and the stratification is stable.
This can also be formulated as an equation of motion for the fluid element with the gradient transformed to an entropy gradient

$$\frac{\partial^2(\Delta r)}{\partial t^2} = -\frac{\mathbf{g}\boldsymbol{\nabla} s}{c_p}\Delta r \tag{20}$$

This differential ratio gives an oscillatory motion with the "Burnt-Vaisala frequency"

$$N^2 = -\frac{g\delta}{H_p}\left(\boldsymbol{\nabla}_{\mathrm{ad}} - \boldsymbol{\nabla} + \frac{\phi}{\delta}\boldsymbol{\nabla}_{\mu}\right) \tag{21}$$

Where

$$\delta = -\frac{\partial \ln\rho}{\partial \ln T}; \quad \phi = \frac{\partial \ln \rho}{\partial \ln \mu}; \quad H_p = -\frac{dr}{d\ln p}; \quad \boldsymbol{\nabla}_{\mathrm{ad}} = \left(\frac{d\ln T}{d\ln p}\right)_{\mathrm{ad}}; \quad \boldsymbol{\nabla} = \left(\frac{d\ln T}{d\ln p}\right)_S; \quad \boldsymbol{\nabla} = \left(\frac{d\ln \mu}{d\ln p}\right)_S$$

are the pressure scale height, adiabatic temperature gradient of the element, the temperature gradient of the surrounding and the chemical gradient of the surrounding. The adiabatic gradient can also be written as $\Delta_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2}$ and the value for the ideal monoatomic gas is 0.4 (what is Γ_2)
Ledoux criterion:

$$\boldsymbol{\nabla}_{\mathrm{rad}} > \boldsymbol{\nabla}_{\mathrm{ad}} + \frac{\phi}{\delta}\boldsymbol{\nabla}_{\mu} \tag{22}$$

which reduces for a uniform composition with $\boldsymbol{\nabla}_{\mu} = 0$ to schwarzschild criterion: $\boldsymbol{\nabla}_{\mathrm{rad}} = \boldsymbol{\nabla}_{\mathrm{ad}}$