# Convection

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## 1 Convection Instability

(AST 462, Eric Blackman's lecture note)
Consider a perfect gas in hydro-static equilibrium in uniform gravity. If $z$ axis is chosen such that gravity is in negative $z$ direction, then $g(z)$ and $\rho(z)$ decreases with $z$. Consider the vertical displacement of the blob as shown Where initially $p$ and $\rho$ have the same density as surroundings, external density and pressure at new position are $p'$ and $\rho'$. Pressure balences inside and outside is maintained swiftly by acoustic waves, but heat imbalence takes longer when mediated by conduction. We can consider the blob to be displaced adiabatically, then let $\rho^*$ be its new density. if $\rho^* < \rho'$, the blob will be bouyant and continue upward, implying instability. if $\rho^* > \rho'$, then the blob will tend to return, making the system stable. So we need to determine $\rho^*/\rho$. For adiabatic flow,

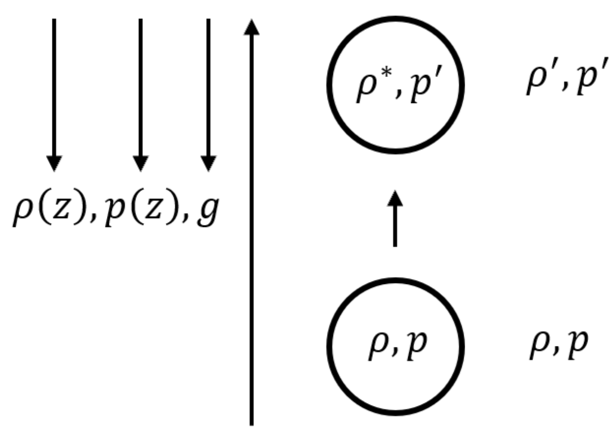
$$\rho^* = \rho\Big(\frac{p'}{p}\Big)^{1/\gamma} \tag{1}$$

If $\frac{dp}{dz}$ is the pressure gradient, then

$$p' = p + \frac{dp}{dz}\Delta z \tag{2}$$

and using

$$\rho^* = \rho\Big(\frac{p'}{p}\Big)^{1/\gamma} = \rho\Big(\frac{p + \frac{dp}{dz}\Delta z}{p}\Big)^{1/\gamma} \approx \rho + \frac{\rho}{\gamma p}\frac{dp}{dz}\Delta z \tag{3}$$



The last step is obtained by expanding to lowest order in $\Delta z$
But for ambient medium,

$$\rho' = \rho + \frac{d\rho}{dz}\Delta z \tag{4}$$

Then, using $\rho = \frac{p}{RT}$

$$\rho' = \rho + \frac{d\rho}{dp}\frac{dp}{dz}\Delta z + \frac{d\rho}{dT}\frac{dT}{dz}\Delta z = \rho + \frac{\rho}{p}\frac{dp}{dz}\Delta z - \frac{\rho}{T}\frac{dT}{dz}\Delta z \tag{5}$$

Using equation 3 and 5, we get

$$\rho^* - \rho' = \frac{\rho}{\gamma p}\frac{dp}{dz}\Delta z - \frac{\rho}{p}\frac{dp}{dz}\Delta z + \frac{\rho}{T}\frac{dT}{dz}\Delta z = \rho\Big[\Big(\frac{1}{\gamma} - 1\Big)\frac{1}{p}\frac{dp}{dz} + \frac{1}{T}\frac{dT}{dz}\Big]\Delta z \tag{6}$$

For stable atmosphere, we require that $\rho^* > \rho'$, thus

$$\Big(\frac{1}{\gamma} - 1\Big)\frac{1}{p}\frac{dp}{dz} + \frac{1}{T}\frac{dT}{dz} > 0 \quad \Rightarrow \quad \frac{dT}{dz} > \Big(1 - \frac{1}{\gamma}\Big)\frac{T}{p}\frac{dp}{dz} \quad \Rightarrow \quad \Big|\frac{dT}{dz}\Big| > \Big(1 - \frac{1}{\gamma}\Big)\frac{T}{p}\Big|\frac{dp}{dz}\Big| \tag{7}$$

The last step because $\frac{dT}{dz} < 0$ and $\frac{dp}{dz} < 0$
This is called the "Schwarzschild stability condition"
Force per unit volume inside the displaced blob is approximatly $(\rho^* - \rho)(-g)$, the equation of motion is therefore

$$\rho^* \frac{d^2\Delta z}{dt^2} = -(\rho^* - \rho')g = -g\rho\Big[\Big(\frac{1}{\gamma} - 1\Big)\frac{1}{p}\frac{dp}{dz} + \frac{1}{T}\frac{dT}{dz}\Big]\Delta z \tag{8}$$

Or equivalently

$$\frac{d^2\Delta z}{dt^2} + N'^2\Delta z = 0 \tag{9}$$

Where

$$N' = \sqrt{\frac{g}{T}\frac{\rho}{\rho^*}\frac{dT}{dz} - \Big(1 - \frac{1}{\gamma}\Big)\frac{g}{p}\frac{\rho}{\rho^*}\frac{dp}{dz}} \tag{10}$$

When $\rho \approx \rho^*$ for lowest order of $\Delta z$, we get

$$N \equiv \sqrt{\frac{g}{T}\frac{dT}{dz} - \Big(1 - \frac{1}{\gamma}\Big)\frac{g}{p}\frac{dp}{dz}} \tag{11}$$

called the "Brunt-Vaisula frequency". For stable equilibrium, the blub will oscillate.
In reality such motion give rise to internal gravity waves by disturbing the surrounding medium. We ignore internal gravity waves by ignoring the effect of the blub motion on external medium. Full treatments account for these waves when full perturbative treatment is developed (?)

# 2 Equations of Stellar Evolution

(Ohlmann Thesis sec. 2.4.1)
In a stationary spherical configuration, the mass profile of the star is given by

$$\frac{\partial m(r,t)}{\partial r} = 4\pi r^2 \rho(r,t) \tag{12}$$

This is a consequence of the geometry. Boundary condition: $m(0) = 0$.
For hydrostatic equilibrium, we need the gravitational acceleration $g = \frac{GM(r)}{r^2}$ for spherical mass distribution. In hydrostatic equilibrium, this self-gravity balences the force from pressure gradient

$$\frac{\partial p(r,t)}{\partial r} = -\frac{Gm(r,t)}{r^2}\rho(r,t) \tag{13}$$

Boundary condition: $p(R) = 0$, where $r = R$ is the surface of the star. This BC can be replaced by a more sophisticated treatment.
To solve this system of equations, an additional relation has to be supplied: the equation of state. The internal structure may not be determined by only considering the mechanical equilibrium, but also the thermal equilibrium that depends on energy sources or sinks and energy transport.
Let $l(r,t)$ be the net energy flux through a sphere of radius $r$, then energy conservation demands

$$\frac{\partial l(r,t)}{\partial r} = 4\pi r^2 \rho(r,t)\big(\epsilon_n - \epsilon_\nu + \epsilon_g\big) \tag{14}$$

where $\epsilon_n$ = nuclear reactions, $\epsilon_\nu$ = neutrino losses and $\epsilon_g$ = mechanical work due to contraction or expansion.
B.C.: At the center of the star $l(0) = 0$; at the surface: $l(R) = L$, the luminosity of the star.
The distribution of the temperature depends on the way energy is transported in the star - either by radiation or convection - and may be written as

$$\frac{\partial T(r,t)}{\partial r} = -\frac{Gm(r,t)}{r^2}\frac{T(r,p)}{p(r,t)}\frac{d\ln T}{d\ln p}\rho(r,t) \tag{15}$$

Is Ohlmann missing by a factor of $\rho$ here? because

$$\frac{\partial T(r,t)}{\partial r} = -\frac{\partial T}{\partial p}\frac{\partial p}{\partial r} = -\frac{Gm}{r^2}\rho\frac{\partial T}{\partial p} = -\frac{Gm\rho}{r^2}\frac{\partial T}{T}\frac{p}{\partial p}\frac{T}{p} = -\frac{Gm\rho}{r^2}\frac{T}{p}\frac{\partial \ln T}{\partial \ln p} \tag{16}$$

where $m = m(r,t)$; $T = T(r,t)$, $p = p(r,t)$
Where $\nabla \equiv \frac{d\ln T}{d\ln p}$ denotes the temperature gradient that depends on energy transportation mechanism. If energy transport by radiation dominates, a diffusion approximation leads to

$$\nabla_{\text{rad}} = \frac{3}{16\pi acG}\frac{\kappa lp}{mT^4} \tag{17}$$

where $a$ = radiation constant, $c$ = speed of light, $\kappa$ = opacity.
If convection is the dominant energy transport mechanism, $\nabla$ has to be supplied by a theory of convection, which is usually the mixing-length theory. In the deep interior it is usually near the adiabatic value $\nabla_{\text{ad}}$.
The chemical composition is given by the mass fractions $X_i(r,t)$ for species $i$, These change due to nuclear reactions and due to mixing in convective regions.

$$\frac{dX_i(r,t)}{dt} = \frac{m_i}{\rho}\Big(\sum_j r_{ji} - \sum_k r_{ik}\Big) \tag{18}$$

Where the derivative is taken in the Lagrangian frame to take into account mixing process. Here $m_i$ denotes the mass of nucleus $i$, and $r_{ij}$ is the reaction rate from nucleus $i$ to nucleus $j$. Moveover, the composition is assumed to be homogeneous in convective regions.
(Ohlmann Thesis sec. 2.4.3)
Most important hydrodynamical instabilities is convective instability caused by buoyancy. Suppose a fluid element embedded in a stellar profile rises adiabatically. If at new radius the buoyancy force is larger compared to the surroundings, the fluid element will rise further; thus ht stratification is unstable. The buoyancy force per unit volume is given by

$$\mathbf{F} = \mathbf{g}\cdot((\Delta\rho)_e - (\Delta\rho)_s)\Delta r\hat{\mathbf{r}} \tag{19}$$

$g$ is gravitational acceleration, the indices denote the quantities in the element (e) and surroundings (s), $\Delta r$ denotes the radial shift. Since the gravity is antiparallel to the density gradient, the difference of the density gradients has to be larger than zero for the right-hand side to be negative; this means that the buoyancy

force points downwards, the fluid element is pulled back, and the stratification is stable.
This can also be formulated as an equation of motion for the fluid element with the gradient transformed to an entropy gradient

$$\frac{\partial^2(\Delta r)}{\partial t^2} = -\frac{\mathbf{g}\boldsymbol{\nabla} s}{c_p}\Delta r \tag{20}$$

This differential ratio gives an oscillatory motion with the "Burnt-Vaisala frequency"

$$N^2 = -\frac{g\delta}{H_p}\left(\boldsymbol{\nabla}_{\mathrm{ad}} - \boldsymbol{\nabla} + \frac{\phi}{\delta}\boldsymbol{\nabla}_\mu\right) \tag{21}$$

Where

$$\delta = -\frac{\partial \ln \rho}{\partial \ln T}; \quad \phi = \frac{\partial \ln \rho}{\partial \ln \mu}; \quad H_p = -\frac{dr}{d\ln p}; \quad \boldsymbol{\nabla}_{\mathrm{ad}} = \left(\frac{d\ln T}{d\ln p}\right)_{\mathrm{ad}}; \quad \boldsymbol{\nabla} = \left(\frac{d\ln T}{d\ln p}\right)_S; \quad \boldsymbol{\nabla} = \left(\frac{d\ln \mu}{d\ln p}\right)_S$$

are the pressure scale height, adiabatic temperature gradient of the element, the temperature gradient of the surrounding and the chemical gradient of the surrounding. The adiabatic gradient can also be written as $\Delta_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2}$ and the value for the ideal monoatomic gas is 0.4 (what is $\Gamma_2$)
Ledoux criterion:

$$\boldsymbol{\nabla}_{\mathrm{rad}} > \boldsymbol{\nabla}_{\mathrm{ad}} + \frac{\phi}{\delta}\boldsymbol{\nabla}_\mu \tag{22}$$

which reduces for a uniform composition with $\boldsymbol{\nabla}_\mu = 0$ to schwarzschild criterion: $\boldsymbol{\nabla}_{\mathrm{rad}} = \boldsymbol{\nabla}_{\mathrm{ad}}$

# 3 OPAL EOS tables for astrophysics Applic. (F. J. Rogers, 1996)

## 3.1 Commonly used EOS

### 3.1.1 Eggleton, Faulkner Flannery (EFF)

Relativistic and accounts for Fermi-Firac statistics for electrons.
Assumes that

1. ions and atoms are in their unperturbed graound state

Ignores that

1. Coulomb interaction

2. treats heavy elements as if they are fully ionized

Advantage:

1. Changes in composition can be easily be accommodated

**CEFF EOS**: Added Debye Coulomb correction

### 3.1.2 Mihalas, Hummer and Dappen (MHD)

Relying on inference from experimental measurements of level shift; used a configurations free energy that depends explicitly on the occupation numbers of the individual states to define an occupation probability.
Assumes that

1. the bound states of atoms and ions are unshifted by the plasma environment

### 3.1.3 OPAL

Start from the grand canonical ensemble (Hill 1956) (a text book), we can get the fundamental, many-body quantum statistical treatments of the EOS of partially ionized plasmas.
The analogous free energy results from the many-body diagrammatic approach, to terms of order $(n_e^2)^{3/2}$ in the coupling parameter is

$$\frac{F}{kT} = -N_e \ln\left(\frac{eg_e}{n_e\mu_e^3}\right) - N_p \ln\left(\frac{eg_p}{n_p\mu_p^3}\right) - N_H \ln\left(\frac{eg_H}{n_H\mu_H^3} Z_{int}^{PL} + \frac{F_{DH}}{kT}\right) \tag{23}$$

where

$$Z_{int}^{PL} = \sum_{nl} (2l+1)(e^{-\beta E_{nl} - 1 + \beta E_{nl}}) \tag{24}$$

is the so-called Planck-Larkin partition function (Rogers 1981, 1986). $Z_{int}^{PL}$ is both finite and a continuous function of temperature and density.
Equation 23 itself is insufficient for precise modeling of hydrogen in solar conditions. For this purpose, *degeneracy, exchange and quantum diffraction corrections* must be considered. The inclusion of these in the pressure of a fully ionized plasma yields

$$\frac{P}{kT} = \frac{n_e I_{3/2}(\alpha_e)}{I_{1/2}(\alpha_e)} + n_p + \frac{P_{ex}}{kT} + \frac{P_{DH}}{kT} f_p(\gamma_{ee}, \gamma_{ei}) \tag{25}$$

Where $I_{n/2}$ functions are the Fermi functions,
$\alpha_e = \frac{\mu_e}{kT}$ is the degeneracy parameter;

$\frac{P_{ex}}{kT}$ is the first-order electron exchange, which is often omitted in astrophysical EOS calculations, but in view of the current need for high precision, it must be included;
$f_p(\gamma_{ee}, \gamma_{ei})$ is a quantum-diffraction correction to the classical Debye-Huckel pressure. (Details in paper).

Summary of OPAL:
The OPAL EOS uses a combination of esact theory and an approximate effetive-potential method to include diffraction corrections.
It includes

1. non-relativistic Fermi-Dirac electrons
2. classical ions
3. all stages of ionization and excitation
4. molecular hydrogen
5. degenerate Columb correction
6. quantum electron diffraction
7. electron exchange
8. pressure ionization
9. terms arising from the so-called ladder diagrams of full quantum theory

It excludes

1. pseudopotential method for going to higher order in electron-electro and electron-ion interaction (as far as in Rogers 1996)

Accurate to the order of

1. Quantum diagrammatic procedure are used to calculate terms to order $(n_e^2)^{5/2}$
2. In the case of hydrogen, it agrees with $(n_e^2)^2$-order correction

## 3.2 Ideal gas EOS (Not in the paper but as a reference)

Gas particles treated as point particles, interacting only through elastic collision.

# 4 OPAL EOS online tables

Resources

- OPAL home
https://opalopacity.llnl.gov/opal.html

- OPAL EOS: calculated with an earlier version of the OPAL equation of state code
https://opalopacity.llnl.gov/pub/opal/eos/

- OPAL EOSPLUS: same as EOS, but denser grid
https://opalopacity.llnl.gov/pub/opal/eosplus/

- OPAL EOS_2001: calculated with an improved version of the OPAL code
https://opalopacity.llnl.gov/Download/

- OPAL EOS_2005: unspecified in online documentation, but according to the readme file, it is an expanded and updated version of EOS_2001 (refer to paper F. J. Rogers and A. Nayfonov, ApJ 2002, 576,1064)
https://opalopacity.llnl.gov/EOS_2005/

There are several interpolation code associated with each EOS version.

# 5 Compare OPAL and Ideal gas EOS

## 5.1 Ideal gas EOS

In astrobear, the independent variables are density $\rho$ and energy $E$. So all our expressions will be in terms of $\rho$ and $E$.

Assume:

$$pV = nkT \tag{26}$$

$$pV^{\gamma} = \text{ constant} \tag{27}$$

where $\gamma = \frac{5}{3}$ for monoatomic gas;

$$E_p = \frac{3}{2}kT \tag{28}$$

where $E_p$ is the mean energy per particle.

Define:
$\mu =$ mean particle mass in hydrogen mass ($\mu = 1$ for pure neutral hydrogen)
$m_s = \mu \cdot m_H$ The mean particle mass
$k =$ Boltzmann constant;
$\mathscr{E} = nE_p =$ mean energy for $n$ particles
$E_g = \frac{\mathscr{E}}{1 \text{ gram}} =$ mean energy for $n$ particles per unit mass.

Derivation:
(Temperature)
By definition,

$$E_g = \frac{\mathscr{E}}{1 \text{ gram}} = \frac{nE_p}{1 \text{ gram}} = \frac{3}{2}nkT\frac{1}{1 \text{ gram}} = \frac{3}{2}kT\frac{1 \text{ gram}}{m_s}\frac{1}{1 \text{ gram}} = \frac{3kT}{2m_s} \tag{29}$$

Thus

$$T = \frac{2m_s}{3k}E_g \tag{30}$$

(Pressure)
Directly from $pV = nkT$, we get

$$p = \frac{nkT}{V} = \frac{MkT}{m_sV} = \frac{\rho kT}{m_s} = \frac{\rho k}{m_s}\frac{2m_s}{3k}E_g = \frac{2\rho E_g}{3} \tag{31}$$

### 5.2 $Z = 0,\ X = 0$ (EOS5_00z0x)

Density range: min: $1 * 10^{-14}$ max: $1 * 10^{7}$
Energy range: $-1.897785 * 10^{13}$ 1.660164e+17
red: OPAL EOS; green: Ideal gas EOS