Convection

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June 2, 2019

1 Convection Instability

(AST 462, Eric Blackman's lecture note)
Consider a perfect gas in hydro-static equilibrium in uniform gravity. If z axis is chosen such that gravity is in negative z direction, then $g(z)$ and $\rho(z)$ decreases with z. Consider the vertical displacement of the blob as shown Where initially p and ρ have the same density as surroundings, external density and pressure at new position are p' and ρ'. Pressure balences inside and outside is maintained swiftly by acoustic waves, but heat imbalence takes longer when mediated by conduction. We can consider the blob to be displaced adiabatically, then let ρ^* be its new density. if $\rho^* < \rho'$, the blob will be bouyant and continue upward, implying instability. if $\rho^* > \rho'$, then the blob will tend to return, making the system stable. So we need to determine ρ^*/ρ. For adiabatic flow,
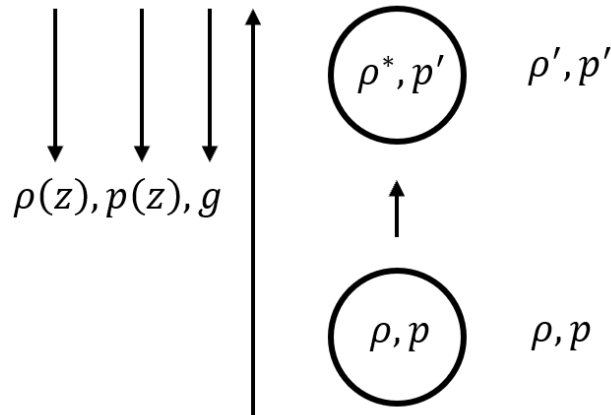
$$\rho^* = \rho\Big(\frac{p'}{p}\Big)^{1/\gamma} \tag{1}$$

If $\frac{dp}{dz}$ is the pressure gradient, then

$$p' = p + \frac{dp}{dz}\Delta z \tag{2}$$

and using

$$\rho^* = \rho\Big(\frac{p'}{p}\Big)^{1/\gamma} = \rho\Big(\frac{p + \frac{dp}{dz}\Delta z}{p}\Big)^{1/\gamma} \approx \rho + \frac{\rho}{\gamma p}\frac{dp}{dz}\Delta z \tag{3}$$

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The last step is obtained by expanding to lowest order in Δz

But for ambient medium,

$$\rho' = \rho + \frac{d\rho}{dz}\Delta z \tag{4}$$

Then, using $\rho = \frac{p}{RT}$

$$\rho' = \rho + \frac{d\rho}{dp}\frac{dp}{dz}\Delta z + \frac{d\rho}{dT}\frac{dT}{dz}\Delta z = \rho + \frac{\rho}{p}\frac{dp}{dz}\Delta z - \frac{\rho}{T}\frac{dT}{dz}\Delta z \tag{5}$$

Using equation 3 and 5, we get

$$\rho^* - \rho' = \frac{\rho}{\gamma p}\frac{dp}{dz}\Delta z - \frac{\rho}{p}\frac{dp}{dz}\Delta z + \frac{\rho}{T}\frac{dT}{dz}\Delta z = \rho\Big[\Big(\frac{1}{\gamma} - 1\Big)\frac{1}{p}\frac{dp}{dz} + \frac{1}{T}\frac{dT}{dz}\Big]\Delta z \tag{6}$$

For stable atmosphere, we require that $\rho^* > \rho'$, thus

$$\Big(\frac{1}{\gamma} - 1\Big)\frac{1}{p}\frac{dp}{dz} + \frac{1}{T}\frac{dT}{dz} > 0 \quad \Rightarrow \quad \frac{dT}{dz} > \Big(1 - \frac{1}{\gamma}\Big)\frac{T}{p}\frac{dp}{dz} \quad \Rightarrow \quad \Big|\frac{dT}{dz}\Big| > \Big(1 - \frac{1}{\gamma}\Big)\frac{T}{p}\Big|\frac{dp}{dz}\Big| \tag{7}$$

The last step because $\frac{dT}{dz} < 0$ and $\frac{dp}{dz} < 0$

This is called the "Schwarzschild stability condition"

Force per unit volume inside the displaced blob is approximatly $(\rho^* - \rho)(-g)$, the equation of motion is therefore

$$\rho^*\frac{d^2\Delta z}{dt^2} = -(\rho^* - \rho')g = -g\rho\Big[\Big(\frac{1}{\gamma} - 1\Big)\frac{1}{p}\frac{dp}{dz} + \frac{1}{T}\frac{dT}{dz}\Big]\Delta z \tag{8}$$

Or equivalently

$$\frac{d^2\Delta z}{dt^2} + N'^2\Delta z = 0 \tag{9}$$

Where

$$N' = \sqrt{\frac{g}{T}\frac{\rho}{\rho^*}\frac{dT}{dz} - \Big(1 - \frac{1}{\gamma}\Big)\frac{g}{p}\frac{\rho}{\rho^*}\frac{dp}{dz}} \tag{10}$$

When $\rho \approx \rho^*$ for lowest order of Δz, we get

$$N \equiv \sqrt{\frac{g}{T}\frac{dT}{dz} - \Big(1 - \frac{1}{\gamma}\Big)\frac{g}{p}\frac{dp}{dz}} \tag{11}$$

called the "Brunt-Vaisula frequency". For stable equilibrium, the blub will oscillate.

In reality such motion give rise to internal gravity waves by disturbing the surrounding medium. We ignore internal gravity waves by ignoring the effect of the blub motion on external medium. Full treatments account for these waves when full perturbative treatment is developed (?)

2   Equations of Stellar Evolution

(Ohlmann Thesis sec. 2.4.1)

In a stationary spherical configuration, the mass profile of the star is given by

$$\frac{\partial m(r,t)}{\partial r} = 4\pi r^2\rho(r,t) \tag{12}$$

This is a consequence of the geometry. Boundary condition: $m(0) = 0$.

For hydrostatic equilibrium, we need the gravitational acceleration $g = \frac{GM(r)}{r^2}$ for spherical mass distribution. In hydrostatic equilibrium, this self-gravity balences the force from pressure gradient

$$\frac{\partial p(r,t)}{\partial r} = -\frac{Gm(r,t)}{r^2}\rho(r,t) \tag{13}$$

Boundary condition: $p(R) = 0$, where $r = R$ is the surface of the star. This BC can be replaced by a more sophisticated treatment.

To solve this system of equations, an additional relation has to be supplied: the equation of state. The internal structure may not be determined by only considering the mechanical equilibrium, but also the thermal equilibrium that depends on energy sources or sinks and energy transport.

Let $l(r, t)$ be the net energy flux through a sphere of radius r, then energy conservation demands

$$\frac{\partial l(r,t)}{\partial r} = 4\pi r^2 \rho(r,t)\big(\epsilon_n - \epsilon_\nu + \epsilon_g\big) \tag{14}$$

where ϵ_n = nuclear reactions, ϵ_ν = neutrino losses and ϵ_g = mechanical work due to contraction or expansion.

B.C.: At the center of the star $l(0) = 0$; at the surface: $l(R) = L$, the luminosity of the star.

The distribution of the temperature depends on the way energy is transported in the star - either by radiation or convection - and may be written as

$$\frac{\partial T(r,t)}{\partial r} = -\frac{Gm(r,t)}{r^2}\frac{T(r,p)}{p(r,t)}\frac{d\ln T}{d\ln p}\rho(r,t) \tag{15}$$

Is Ohlmann missing by a factor of ρ here? because

$$\frac{\partial T(r,t)}{\partial r} = -\frac{\partial T}{\partial p}\frac{\partial p}{\partial r} = -\frac{Gm}{r^2}\rho\frac{\partial T}{\partial p} = -\frac{Gm\rho}{r^2}\frac{\partial T}{T}\frac{p}{\partial p}\frac{T}{p} = -\frac{Gm\rho}{r^2}\frac{T}{p}\frac{\partial \ln T}{\partial \ln p} \tag{16}$$

where $m = m(r,t)$; $T = T(r,t)$, $p = p(r,t)$

Where $\boldsymbol{\nabla} \equiv \frac{d\ln T}{d\ln p}$ denotes the temperature gradient that depends on energy transportation mechanism. If energy transport by radiation dominates, a diffusion approximation leads to

$$\boldsymbol{\nabla}_{\text{rad}} = \frac{3}{16\pi acG}\frac{\kappa l p}{m T^4} \tag{17}$$

where a = radiation constant, c = speed of light, κ = opacity.

If convection is the dominant energy transport mechanism, $\boldsymbol{\nabla}$ has to be supplied by a theory of convection, which is usually the mixing-length theory. In the deep interior it is usually near the adiabatic value $\boldsymbol{\nabla}_{\text{ad}}$.

The chemical composition is given by the mass fractions $X_i(r,t)$ for species i, These change due to nuclear reactions and due to mixing in convective regions.

$$\frac{dX_i(r,t)}{dt} = \frac{m_i}{\rho}\bigg(\sum_j r_{ji} - \sum_k r_{ik}\bigg) \tag{18}$$

Where the derivative is taken in the Lagrangian frame to take into account mixing process. Here m_i denotes the mass of nucleus i, and r_{ij} is the reaction rate from nucleus i to nucleus j. Moveover, the composition is assumed to be homogeneous in convective regions.

(Ohlmann Thesis sec. 2.4.3)

Most important hydrodynamical instabilities is convective instability caused by buoyancy. Suppose a fluid element embedded in a stellar profile rises adiabatically. If at new radius the buoyancy force is larger compared to the surroundings, the fluid element will rise further; thus ht stratification is unstable. The buoyancy force per unit volume is given by

$$\mathbf{F} = \mathbf{g} \cdot ((\Delta\rho)_e - (\Delta\rho)_s)\Delta r\hat{\mathbf{r}} \tag{19}$$

g is gravitational acceleration, the indices denote the quantities in the element (e) and surroundings (s), Δr denotes the radial shift. Since the gravity is antiparallel to the density gradient, the difference of the density gradients has to be larger than zero for the right-hand side to be negative; this means that the buoyancy

force points downwards, the fluid element is pulled back, and the stratification is stable.

This can also be formulated as an equation of motion for the fluid element with the gradient transformed to an entropy gradient

$$\frac{\partial^2 (\Delta r)}{\partial t^2} = -\frac{\mathbf{g}\boldsymbol{\nabla} s}{c_p} \Delta r \tag{20}$$

This differential ratio gives an oscillatory motion with the "Burnt-Vaisala frequency"

$$N^2 = -\frac{g\delta}{H_p}\left(\boldsymbol{\nabla}_{\mathrm{ad}} - \boldsymbol{\nabla} + \frac{\phi}{\delta}\boldsymbol{\nabla}_\mu\right) \tag{21}$$

Where

$$\delta = -\frac{\partial \ln \rho}{\partial \ln T}; \quad \phi = \frac{\partial \ln \rho}{\partial \ln \mu}; \quad H_p = -\frac{dr}{d\ln p}; \quad \boldsymbol{\nabla}_{\mathrm{ad}} = \left(\frac{d\ln T}{d\ln p}\right)_{\mathrm{ad}}; \quad \boldsymbol{\nabla} = \left(\frac{d\ln T}{d\ln p}\right)_S; \quad \boldsymbol{\nabla} = \left(\frac{d\ln \mu}{d\ln p}\right)_S$$

are the pressure scale height, adiabatic temperature gradient of the element, the temperature gradient of the surrounding and the chemical gradient of the surrounding. The adiabatic gradient can also be written as $\Delta_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2}$ and the value for the ideal monoatomic gas is 0.4 (what is Γ_2)

Ledoux criterion:

$$\boldsymbol{\nabla}_{\mathrm{rad}} > \boldsymbol{\nabla}_{\mathrm{ad}} + \frac{\phi}{\delta}\boldsymbol{\nabla}_\mu \tag{22}$$

which reduces for a uniform composition with $\boldsymbol{\nabla}_\mu = 0$ to schwarzschild criterion: $\boldsymbol{\nabla}_{\mathrm{rad}} = \boldsymbol{\nabla}_{\mathrm{ad}}$

3   Statistical Mechanics (R. K. Pathria)

3.1   Chapter 2: Ensemble theory

Macrostate: (N, V, E)
Phase space of a classical system

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{23}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{24}$$

where H is the Hamiltonian of the system
The overall picture of the movement in the phase space p, q is best illustrated by a "density function" $\rho(p, q, t)$
An ensemble is said to be stationary if

$$\frac{\partial \rho}{\partial t} = 0 \tag{25}$$

Liouville's theorem (derivation pg 32, chapter 2.2)
Assumes in equilibrium

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + [\rho, H] = 0 \tag{26}$$

For the preceding equation and equation 25 to be true, we require that

$$[\rho, H] = \sum_{i=1}^{3N} \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = 0 \tag{27}$$

One way to satisfy the preceding requirement is $\rho(p, q) = \text{constant}$.
A more general way to satisfy the requirement is to assume that the dependence of ρ on (p, q) comes only through an explicit dependence on Hamiltonian. That is $\rho(q, p) = \rho[H(q, p)]$. A more natural choice is

$$\rho(q, p) \propto e^{-\frac{H(q,p)}{kT}} \tag{28}$$

The ensemble so defined is referred as the *canonical ensemble*
Microcanonical ensemble
The microcanonical ensemble is a collection of systems for which the density function ρ is, at all times, given by

$$\left. \begin{array}{l} \rho(q, p) = \text{ constant} \quad \text{if } (E - \frac{1}{2}\Delta \leq H(q, p) \leq (E + \frac{1}{2}\Delta) \\ \qquad\qquad 0 \qquad\qquad\qquad\qquad \text{otherwise} \end{array} \right\} \tag{29}$$

The volume ω (of the allowed region of the phase space) is a direct measure of the multiplicity Γ of the microstates accessible to the system. To estabilsh a connection between ω and Γ, we define a fundamental volume ω_0 which can be regarded as "equivalent to one microstate". We therefore have

$$\Gamma = \frac{\omega}{\omega_0} \tag{30}$$

Thus

$$S = k \ln \Gamma = k \ln \frac{\omega}{\omega_0} \tag{31}$$

3.2   Chapter 3: Canonical ensemble

One good way to describe a system is in terms of its temperature. Consider a system in energy change with a heat reservoir. If the heat reservoir consists of infinitely large number of mental copies of the given system, we get an ensemble of systems. This is an ensemble defined through the parameters N, V and T. Such an ensemble is referred to as a _canonical ensemble_.

In the canonical ensemble, the energy E is a varibale. The question arises: what is the probability that at any time t, a system in the ensemble is found to be in one of the states characterized by the energy value E_r?

Let the heat reservoir have energy E_r', Let the number of these states be denoted by $\Omega'(E_r')$. Note that this function depends on the nature of the reservoir. The system A have characterization energy E_r such that

$$E_r + E_r' = E^{(0)} = \text{ constant} \tag{32}$$

Consider the probability distribution, expanding at $E^{(0)}$

$$P_r \propto \Omega'(E_r') \equiv \Omega'(E^{(0)} - E_r) \tag{33}$$

$$\ln P_r = \ln \Omega'(E^{(0)}) + \left(\frac{\partial \ln \Omega'}{\partial E'}\right)_{E'=E^{(0)}} (E_r' - E^{(0)}) + ... \approx \text{ const } - \beta' E_r \tag{34}$$

In equilibrium, $\beta' = \beta = 1/kT$, Thus

$$P_r = \frac{e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} \tag{35}$$

3.2.1   A system in the canaical ensemble

Consider an ensemble of $\mathcal{N}$ identical systems, sharing a total energy $\mathcal{E}$. Let n_r denotes the number of systems which, at any time t, have the energy value E_r. Then the following conditions have the be satisfied

$$\begin{aligned} \sum_r n_r &= \mathcal{N} \\ \sum_r E_r n_r &= \mathcal{E} = \mathcal{N}U \end{aligned} \tag{36}$$

where U denotes average energy per system in the ensemble.

Let $\{n^*\}$ be the most probable distribution set; expectation values be $\langle n_r \rangle$.

Denote the number of different ways to obtain a state of an ensemble

$$W(n_r) = \frac{\mathcal{N}!}{n_0! n_1! n_2!...} \tag{37}$$

Now determine $\{n^*\}$ and $\langle n_r \rangle$. In principle they should be the same.

most probable values method:

$$\ln W = \ln(\mathcal{N}!) - \sum \ln(n_r!) \approx \mathcal{N} \ln \mathcal{N} - \sum n_r \ln n_r \tag{38}$$

Thus, shift n_r to $n_r + \delta n_r$

$$\delta \ln W = -\sum_r (\ln n_r + 1)\delta n_r \tag{39}$$

Since n_r is at maximum, $\delta \ln W = 0$, $\sum_r \delta n_r = 0$ and $\sum_r E_r \delta n_r = 0$. Therefore, using Lagrange multiplier [1], we get

$$\sum_r [-\ln(n_r^* + 1) - \alpha - \beta E_r]\delta n_r = 0 \tag{40}$$

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[1]Maximize function f, subject to constraint function $g = 0$, we have $f - \lambda g = 0$, where λ is some parameter.

Since n_r is arbitrary, the only way for this equation to hold is

$$\ln n_r^* = -(\alpha + 1) - \beta E_r \tag{41}$$

Therefore

$$n_r^* = C e^{-\beta E_r} \tag{42}$$

Where C is some undetermined factor, now take the ratio

$$\frac{n_r}{\mathcal{N}} = \frac{e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} \tag{43}$$

And we can get β by solving

$$\frac{\mathcal{E}}{\mathcal{N}} = \frac{\sum_r E_r e^{-\beta E_r}}{\sum_r e^{-\beta E_r}} \tag{44}$$

4 OPAL EOS tables for astrophysics Applic. (F. J. Rogers, 1996)

4.1 Commonly used EOS

4.1.1 Eggleton, Faulkner Flannery (EFF)

Relativistic and accounts for Fermi-Firac statistics for electrons.
Assumes that

1. ions and atoms are in their unperturbed graound state

Ignores that

1. Coulomb interaction

2. treats heavy elements as if they are fully ionized

Advantage:

1. Changes in composition can be easily be accommodated

CEFF EOS: Added Debye Coulomb correction

4.1.2 Mihalas, Hummer and Dappen (MHD)

Relying on inference from experimental measurements of level shift; used a configurations free energy that depends explicitly on the occupation numbers of the individual states to define an occupation probability.
Assumes that

1. the bound states of atoms and ions are unshifted by the plasma environment

4.1.3 OPAL

Start from the grand canonical ensemble (Hill 1956) (a text book), we can get the fundamental, many-body quantum statistical treatments of the EOS of partially ionized plasmas.
The analogous free energy results from the many-body diagrammatic approach, to terms of order $(n_e^2)^{3/2}$ in the coupling parameter is

$$\frac{F}{kT} = -N_e \ln\left(\frac{eg_e}{n_e\mu_e^3}\right) - N_p \ln\left(\frac{eg_p}{n_p\mu_p^3}\right) - N_H \ln\left(\frac{eg_H}{n_H\mu_H^3}Z_{int}^{PL} + \frac{F_{DH}}{kT}\right) \tag{45}$$

where

$$Z_{int}^{PL} = \sum_{nl}(2l+1)(e^{-\beta E_{nl}-1+\beta E_{nl}}) \tag{46}$$

is the so-called Planck-Larkin partition function (Rogers 1981, 1986). Z_{int}^{PL} is both finite and a continuous function of temperature and density.
Equation 45 itself is insufficient for precise modeling of hydrogen in solar conditions. For this purpose, *degeneracy, exchange and quantum diffraction corrections* must be considered. The inclusion of these in the pressure of a fully ionized plasma yields

$$\frac{P}{kT} = \frac{n_e I_{3/2}(\alpha_e)}{I_{1/2}(\alpha_e)} + n_p + \frac{P_{ex}}{kT} + \frac{P_{DH}}{kT}f_p(\gamma_{ee}, \gamma_{ei}) \tag{47}$$

Where $I_{n/2}$ functions are the Fermi functions,
$\alpha_e = \frac{\mu_e}{kT}$ is the degeneracy parameter;

$\frac{P_{ex}}{kT}$ is the first-order electron exchange, which is often omitted in astrophysical EOS calculations, but in view of the current need for high precision, it must be included;

$f_p(\gamma_{ee}, \gamma_{ei})$ is a quantum-diffraction correction to the classical Debye-Huckel pressure. (Details in paper).

Summary of OPAL:

The OPAL EOS uses a combination of esact theory and an approximate effetive-potential method to include diffraction corrections.

It includes

1. non-relativistic Fermi-Dirac electrons

2. classical ions

3. all stages of ionization and excitation

4. molecular hydrogen

5. degenerate Columb correction

6. quantum electron diffraction

7. electron exchange

8. pressure ionization

9. terms arising from the so-called ladder diagrams of full quantum theory

It excludes

1. pseudopotential method for going to higher order in electron-electro and electron-ion interaction (as far as in Rogers 1996)

Accurate to the order of

1. Quantum diagrammatic procedure are used to calculate terms to order $(n_e^2)^{5/2}$

2. In the case of hydrogen, it agrees with $(n_e^2)^2$-order correction

Note

Energy can be negative because OPAL EOS also considers potential energy, such as electron-ion potential $(H^-, H_2^+, H_2, He_2^+$. See Rogers & Nayfonov 2002 figure 1 for more details)

4.2   Ideal gas EOS (Not in the paper but as a reference)

Gas particles treated as point particles, interacting only through elastic collision.

5   OPAL EOS online tables

Resources

- OPAL home
https://opalopacity.llnl.gov/opal.html

- OPAL EOS: calculated with an earlier version of the OPAL equation of state code
https://opalopacity.llnl.gov/pub/opal/eos/

- OPAL EOSPLUS: same as EOS, but denser grid
https://opalopacity.llnl.gov/pub/opal/eosplus/

- OPAL EOS_2001: calculated with an improved version of the OPAL code
https://opalopacity.llnl.gov/Download/

- OPAL EOS_2005: unspectified in online documentation, but according to the readme file, it is an expanded and updated version of EOS_2001 (refer to paper F. J. Rogers and A. Nayfonov, ApJ 2002, 576,1064)
https://opalopacity.llnl.gov/EOS_2005/

There are several interpolation code associated with each EOS version.

6   Compare OPAL and Ideal gas EOS

6.1   Ideal gas EOS

In astrobear, the independent variables are density ρ and energy E. So all our expressions will be in terms of ρ and E.

Assume:

$$pV = nkT \tag{48}$$

$$pV^{\gamma} = \text{ constant} \tag{49}$$

where $\gamma = \frac{5}{3}$ for monoatomic gas;

$$E_p = \frac{3}{2}kT \tag{50}$$

where E_p is the mean energy per particle.

Define:
μ = mean particle mass in hydrogen mass ($\mu = 1$ for pure neutral hydrogen)
$m_s = \mu \cdot m_H$ The mean particle mass
k = Boltzmann constant;
$\mathscr{E} = nE_p$ = mean energy for n particles
$E_g = \frac{\mathscr{E}}{1 \text{ gram}}$ = mean energy for n particles per unit mass.

Derivation:
(Temperature)
By definition,

$$E_g = \frac{\mathscr{E}}{1 \text{ gram}} = \frac{nE_p}{1 \text{ gram}} = \frac{3}{2}nkT\frac{1}{1 \text{ gram}} = \frac{3}{2}kT\frac{1 \text{ gram}}{m_s}\frac{1}{1 \text{ gram}} = \frac{3kT}{2m_s} \tag{51}$$

Thus

$$T = \frac{2m_s}{3k}E_g \tag{52}$$

(Pressure)
Directly from $pV = nkT$, we get

$$p = \frac{nkT}{V} = \frac{MkT}{m_sV} = \frac{\rho kT}{m_s} = \frac{\rho k}{m_s}\frac{2m_s}{3k}E_g = \frac{2\rho E_g}{3} \tag{53}$$

6.1.1   To make 2D comparison plot: Dependent variable: E_g

We try to find, for any variable x, a function $E_g(\rho, x)$ to plot ideal gas EOS in 2D
From above, we can get pressure p

$$E_g(\rho, p) = \frac{3kT}{2m_s} = \frac{3k}{2m_s}\frac{pV}{nk} = \frac{3pV}{2nm_s} = \frac{3pV}{2m} = \frac{3p}{2\rho} \tag{54}$$

Temperature T

$$E_g(\rho, T) = \frac{3kT}{2m_s} \tag{55}$$

6.2 $Z = 0$, $X = 0$ (EOS5_00z0x)

Density range: min: $1 * 10^{-14}$ max: $1 * 10^{7}$
Energy range: $-1.897785 * 10^{13}$ 1.660164e+17
red: OPAL EOS; green: Ideal gas EOS