Equation-of-State Challenges

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Enter: the equation of state

- Equation-of-state diagnosis possible, because

Acoustic modes (largely) governed by the adiabatic sound speed

$$c^2 = \frac{p}{\rho}\,\gamma_1$$

$$\gamma_1 = \left(\frac{\partial p}{\partial \rho}\right)_S$$ (often denoted Γ_1)

Fortunate situation

- In convection zone, the Sun is (largely) adiabatically stratified, its structure is mainly determined by thermodynamics.
- Little “contamination” from opacity
- Helioseismology can probe locally

History of the stellar equation of state

- In stellar physics, before 1975, normal (non-degenerate) stars were successfully modeled by

$$pV = (\Sigma_i N_i)kT$$

- With N_i from a Saha equation

This is good to 90% accuracy!

Early helioseismology

- From 1975-1985, more refined equations of state, mainly

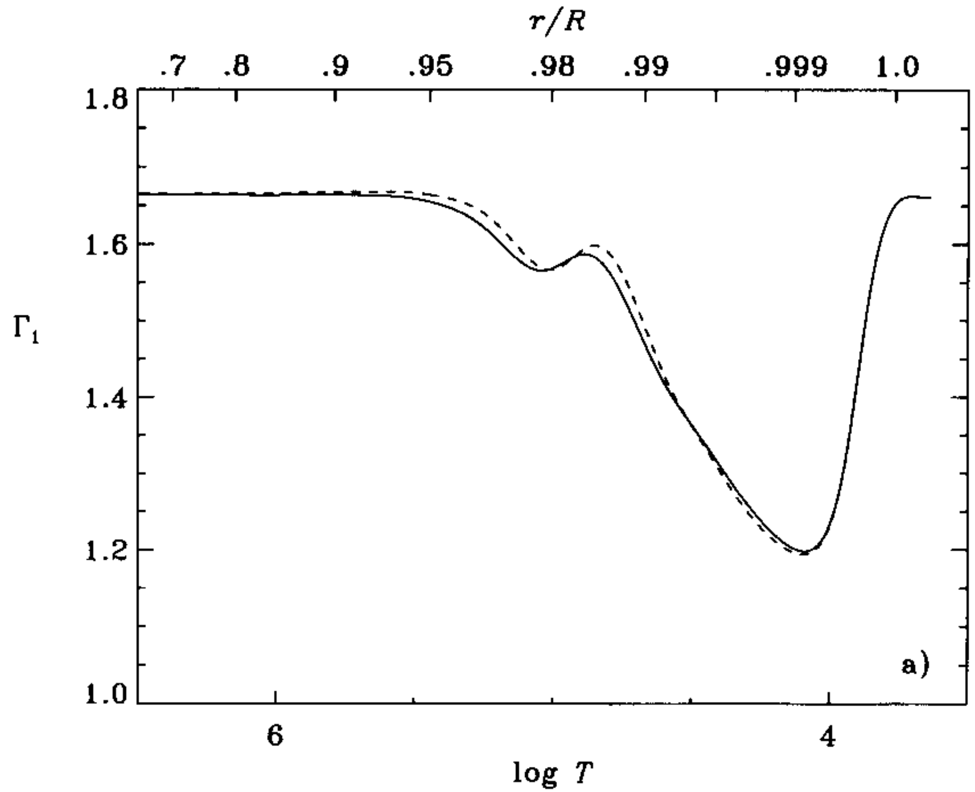
- Detailed chemical composition
- Fermi-Dirac electrons
- Debye-Hückel screening

good to 95-99% accuracy!

Two similar solar models…

- Both identical, other than their equations of state. One is with Debye-Hückel screening, one without. Their adiabatic exponents are:

Dashed: with screening
Solid: without screening



From: Christensen-Dalsgaard & Däppen 1992, A&A Rev. **4**, 267

Two main approaches: introduction

- Free-energy minimization
  chemical picture
  intuitive, but highly practical

- Grand-canonical expansions
  Physical picture
  systematic method for non-ideal corrections

Chemical picture

- Reactions $(\mathrm{H} \leftrightarrow \mathrm{H}^{+} + \mathrm{e}^{-} \;,\; \text{etc.})$
- Constraints $(N_{\mathrm{H}} + N_{\mathrm{p}} = \text{const.} \;,\; \text{etc.})$
- Minimize $F(T, V, N_{\mathrm{H}}, N_{\mathrm{p}}, N_{\mathrm{e}^{-}}, \ldots)$

  !!!subject to constraints!!!
- In practice, write (intuition!)

$$F_{\mathrm{tot}} = F_{\mathrm{nuc}} + F_{\mathrm{e}} + F_{\mathrm{interactions}} + \ldots$$

- Consistent $p = -\left(\frac{\partial F}{\partial V}\right)_T \;,\; \text{etc.}$

MHD

- Fairly conventional realization in chemical picture
- Key ingredient: occupation probabilities

Hummer, D.G. & Mihalas, D.M. 1988, *ApJ* **331**, 794;
Mihalas, D.M., Däppen, W. & Hummer, D.G. 1988, *ApJ* **331**, 815
Däppen,W., Mihalas, D.M., Hummer, D.G. & Mihalas, B.W. 1988, *ApJ* **332**, 261

$$Z_{jk}^{\mathrm{int}} = \sum_i w_{ijk} \; g_{ijk} \exp\left(-\beta E_{ijk}\right)$$

$$(w_{ijk})_{\mathrm{neutral}} = \exp\left[-(4\pi/3V)\sum_{j',k'} N_{j'k'}(r_{ijk}+r_{1j'k'})^3\right]$$

$$(w_{ijk})_{\mathrm{charged}} = \exp\left\{-\left(\frac{4\pi}{3V}\right) 16 \left[\frac{(Z_{jk}+1)^{1/2} e^2}{K_{ijk}^{1/2}\chi_{ijk}}\right]^3 \sum_{j',k'} N_{j'k'} Z_{j'k'}^{3/2}\right\}$$

ACTEX (OPAL)

- First successful stellar modeling with an equation of state in the physical picture

  Rogers, F.J. 1986, *ApJ* **310**, 723;
  Rogers, F.J., Swenson, F.J. & Iglesias, C.A. 1996, *ApJ* **456**, 902
  Rogers, F.J. & Nayfonov, A. 2002, *ApJ* **576**, 1064

- Key points: systematic expansions (*z* = activity)

$$\frac{p}{k_B T} = z + z^2 b_2 + z^3 b_3 + \ldots \; ; \quad \rho = \frac{z}{k_B T}\left(\frac{\partial p}{\partial z}\right)$$

Planck-Larkin Partition Function

$$\mathrm{PLPF} = \sum_{nl}(2l+1)\left[\exp\left(-\frac{E_{nl}}{kT}\right) - 1 + \frac{E_{nl}}{k_B T}\right]$$

Classification of EOS models

	Chemical picture	Physical picture
Representative model	MHD EOS	OPAL EOS
Basic characteristics	Assumes the notion of "atoms" and "molecules" Treats the ionization process like a chemical reaction; Assumes Modularity of partition function; Thermodynamic equilibrium is achieved by free energy minimization method (FEMM)	Only consists of fundamental particles "electrons" and "protons", the notion of "composite" particles arise naturally within the formalism Applies systematic grand canonical approach; Uses an activity series expansion (ACTEX)

Strength and weakness

	MHD EOS	OPAL EOS
Strength	Heuristic, flexible, easier to interpret due to its modularity; open-source, easy to implement, thus more suitable to astrophysical needs	Systematic, rigorous due to its theoretical foundation; no enforced assertions like modularity
Weakness	Unsystematic, sometimes inconsistent, inherently incapable to go to higher orders	Only exists in tabular form for fixed chemical compositions; code is proprietary; formalism becomes unwieldy going to higher order, not suitable for computations of complex astrophysical compositions

First solar result: we saw that…

- …the static screened Coulomb potential

$$V_{r_\mathrm{D}} = -\frac{Q_1 Q_2}{r} \exp^{-r/r_\mathrm{D}}$$

describes the dominant nonideal effect

Christensen-Dalsgaard, J., Däppen, W. & Lebreton, Y. 1988, *Nature*, **336**, 634

- Modeling is without problems, both in chemical and physical picture

At next smaller level…

- Various smaller competing effects:

- Population of excited states
- Diffraction and exchange terms
- Parametric "size" in hard-spheres
- Relativistic correction for electrons

  etc.

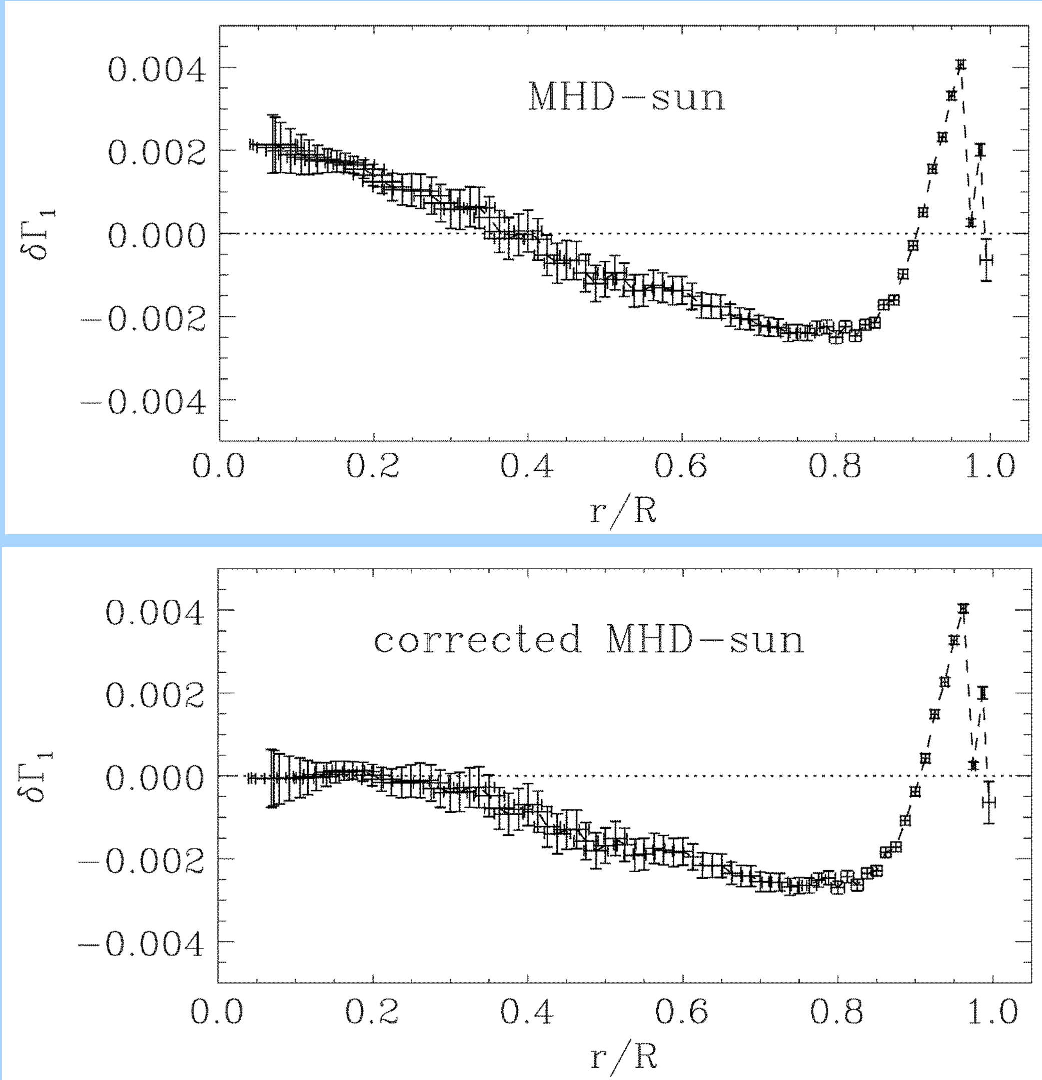
Relativistic electrons in the Sun

- Relativistic corrections are expected to be small, central temperature

$$kT \approx 1 \text{ keV} \ll 511 \text{ keV}$$

- And yet: the effect can be observed!!
  (Elliot & Kosovichev 1998, ApJ, **500** L199)

Models with and without relativistic electrons



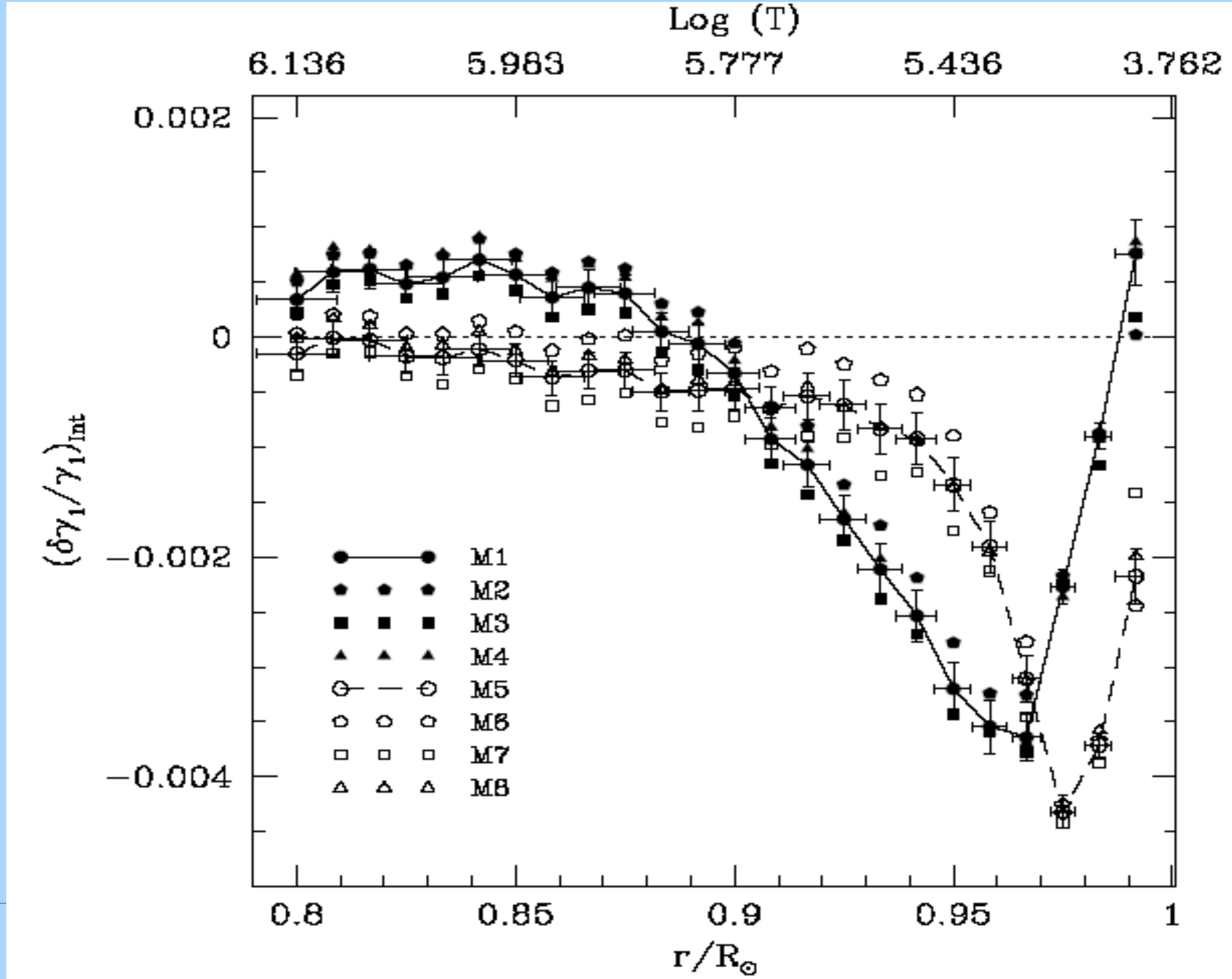
Figures from:

Elliot, J.R. & Kosovichev, A.G.

1998, *ApJ*, **500** L199

Inversions for γ_1 (Sun-model)

- ● Filled (1-4): chemical picture (MHD)
- O Open (5-8): physical picture (ACTEX)



Details in:
S. **Basu**,
W. Däppen,
A. Nayfonov, 1999
ApJ, **518**: 985

Inversions for c^2 (Sun-model)

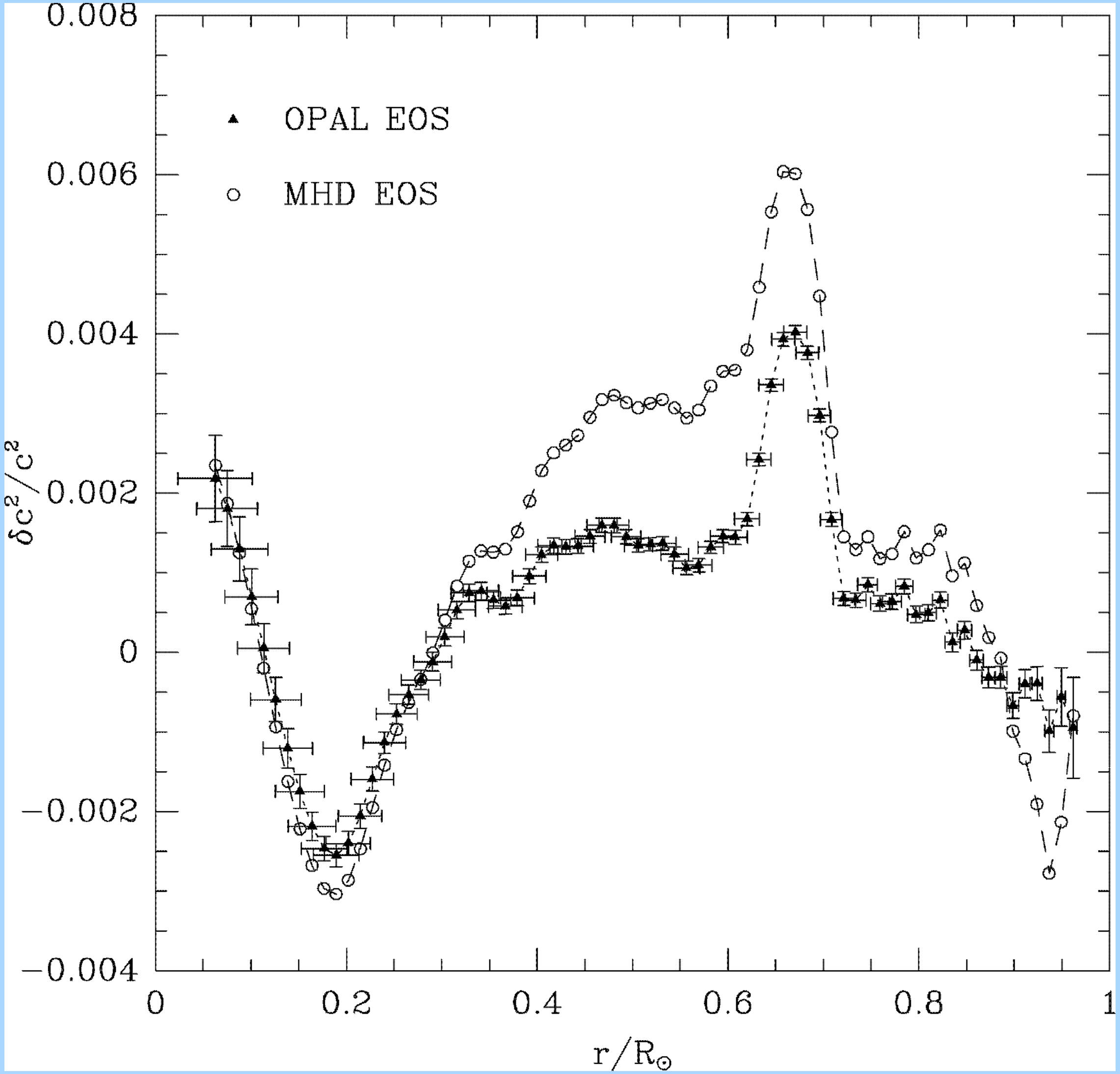


Figure from:
S. **Basu**

OPAL fares better than MHD…

- Why? Likely answer:

- There is no PLPF in MHD
- There are no scattering states in MHD

- Open question: is it fundamentally impossible to find PLPF entirely from within the chemical picture?

Tentative answer…

… to the open question:

Well, perhaps one can, like if one wished to fix the Ptolemaic system by using 80,000 parameters instead of the 80 that Ptolemy himself used!

The alternative

Even before a consistent solution is found, use results from the physical picture (which had, of course, been suggested many times before, *e.g.* by Ebeling, Rogers, Starostin).

Aihua Liang (2005, PhD Thesis, USC)

- OPAL simulator for a quantitative study, which brings the OPAL program to public domain for the first time

  (so far H only, other elements soon)
- Incorporating scattering-state terms and PLPF into MHD
- Comparing thermodynamic quantities of the modified MHD formalism with OPAL results

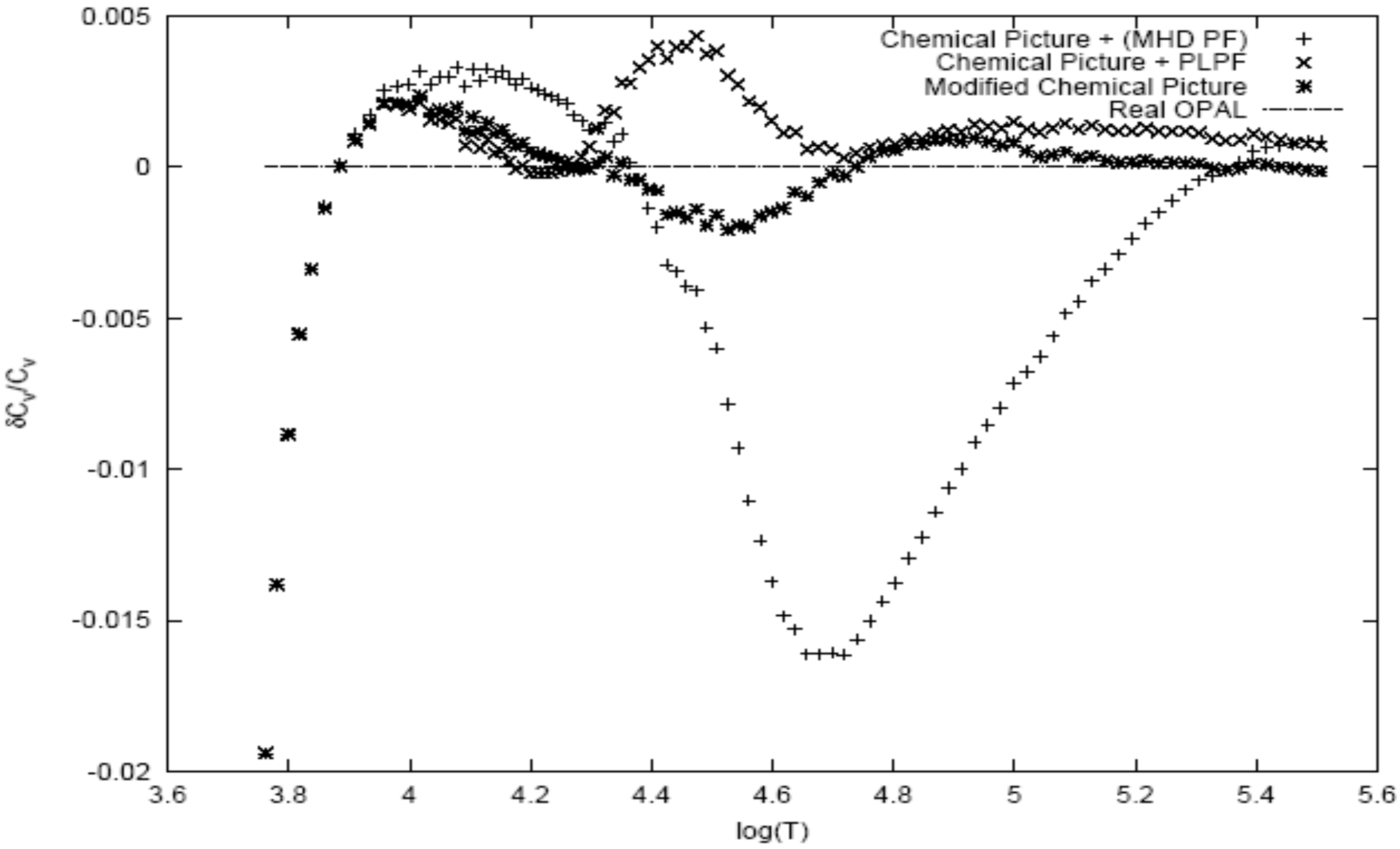
Numerical results:

Three cases, all being compared with the real OPAL data tables:

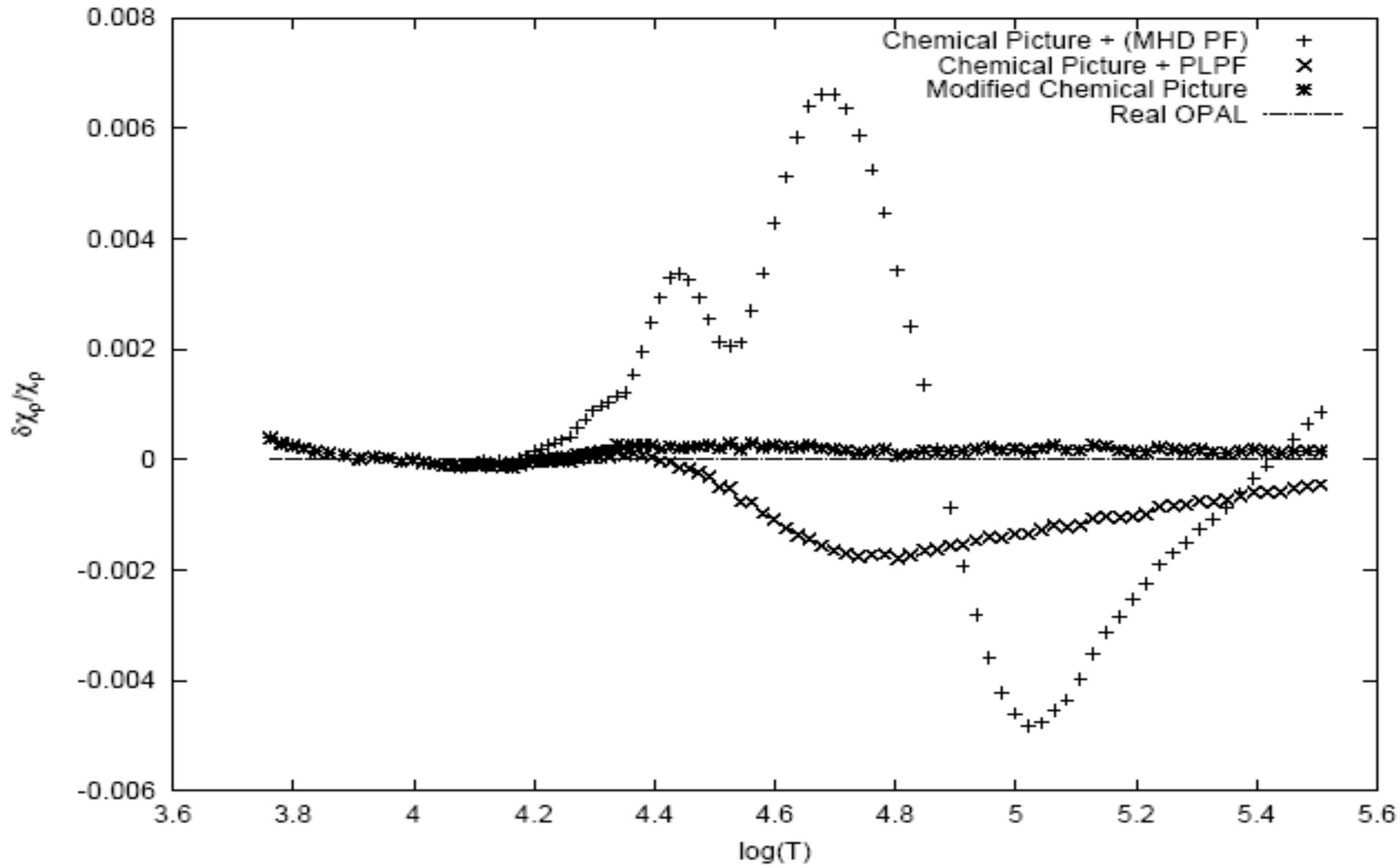
- Chemical picture + PLPF, but no scattering terms are taken into account.
- Chemical picture + original MHD partition function
- Modified chemical picture (PLPF + scattering terms)

- $C_v = T(\partial S/\partial T)_v;$
- $\chi_\varrho = (\partial \ln p/\partial \ln \varrho)_T;$
- $\chi_T = (\partial \ln p/\partial \ln T)_\varrho;$

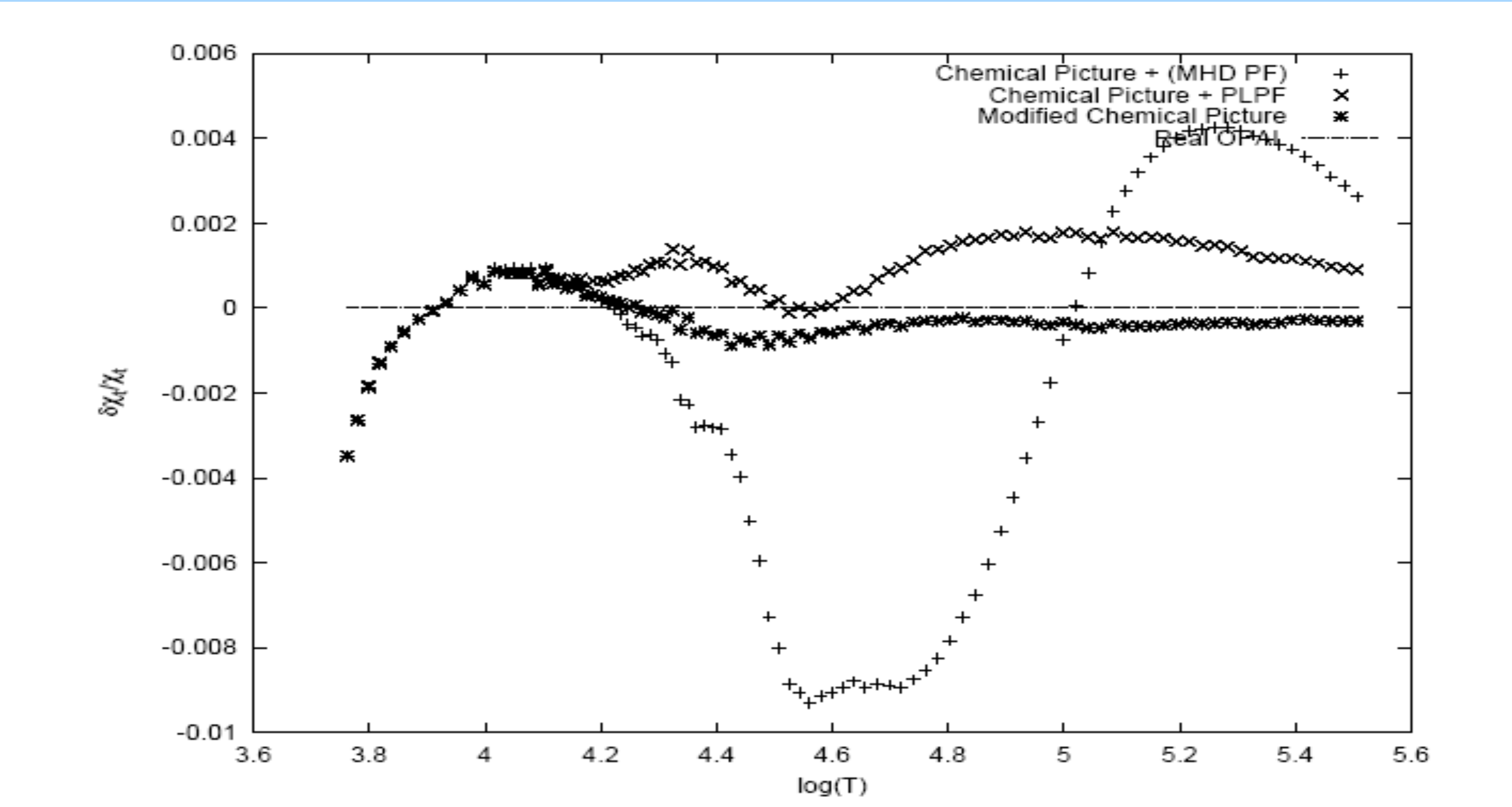
Numerical results:



Numerical results:



Numerical results:



Outlook & conclusions

- So far, no equation of state has successfully matched the solar data to the observed accuracy
- Efforts beyond the current level are therefore warranted
- Among current open physical issues are the mechanism of pressure ionization, and the development to higher-order terms (such as exchange and diffraction)
- Although the solar plasma is only weakly coupled, the new expansion coefficients will be useful for more strongly-coupled Coulomb systems
- A public-domain version of the ACTEX (OPAL) code will be useful for, *e.g.*, *in-situ* calculations of stellar models, without recourse to interpolation in thermodynamic tables
- Phenomenological equations of state can be parameterized to match the solar data exactly